On the Trisection of Archangels

The title can be thought of as a mnemonic for the point of this essay, which only becomes apparent near the end. For those unfamiliar with Euclid, my encounter with the problem of trisecting an angle will be educational. For geometers and educated smarty-pants, it should be amusing. But no matter who you are, I can promise you at least one thing: that, after reading this essay, you will be able to trisect any angle with only a ruler and compass, in the canonical Euclidean way. Further, you will be able to use this unassailable proof in order to amuse your friends.

Rather than put all my results on a page or three in a strictly mathematical way, I will turn my experience into a narrative for the benefit of other free-range mathematicians. This story may be a consolation for them. So here is ...

How It All Began

Like many people who have degrees in mathematics, I managed somehow to miss Euclid almost entirely. But I was led back to him by Gauss and Lagrange, Newton and De Morgan, all of whom display Euclid's power. So I came to Euclid late. Late enough, that I already knew about the constructions which were impossible to solve: trisection of an angle and quadrature of a circle. You will see by the end of this essay that these are very similar problems in a way you might not have thought of.

As I worked through Euclid, it was my habit to use anything trisected for a mental exercise. Something is trisected. You hold it up to an arbitrary angle. And you ask why this does not trisect every angle. And for some time, there was always an obvious answer. That is, until my encounter with a trapezium. A trapezium is a 4-gon with two parallel sides. And its diagonals cut each other in the ratio of their bases (those parallel sides). So if one base is twice the other, the diagonals intersect at their mutual point of trisection. I held this up to my arbitrary angle. And it trisected the angle's chord very nicely, thank you.

Which was annoying because I was working on something else and hadn't expected to have to actually think about why the angle was not trisected. But I put my work aside. I drew a nice picture of what I had so far. And I asked myself, What is not trisected here?

My first thought was that each chord would have its points of trisection on a different trisecting line. In that case, an infinite series of chords would produce an infinitely long curve of segmented lines. Or that was how I pictured it. A quick construction showed me that was not the case. The same lines, produced, trisected all chords across the angle. I should mention that the trapezium appeared at Euclid VI 2. And after making my little construction, it occurred to me that VI 2 already told me that parallel bases retained the proportions of my original isosceles triangle.

So, maybe, the triangle itself was not actually trisected. But cut it in half with the median from the vertex, lay it on its side, and the right triangles staring you in the face scream their trisection to the skies.

I used some bad language at this point. Then, because nothing else came to my tiny mind as trisectible, I set it aside for the day. It wasn't until the next day that I experienced ...

The Failure of Greek Engineering

Those of you more knowledgeable or less slow than I am have been waiting for this part. The next day, I knew that the problem had to be in what happened when you split the angle and move the sub-chords up into the new sub-arcs, so to speak. I am describing things as picturesquely as possible because I am too lazy to make the necessary diagrams. But the diagrams would be simple. So it won't hurt you to make them in your head. The exercise will actually be good for you. Really.

Again, I made a construction of the problem and attempted, in pure Euclidean fashion, to "move up the sub-chords." I quickly learned why the Greeks were unable to invent the steam engine or the cuckoo clock. There is absolutely no accuracy in a unmarked ruler and a compass to speak of. You are not so much constructing things as you are making neat drawings which are suggestive of possible future engineering ideals. The Greeks were in fact lucky to be able to produce a wooden horse.

In short, I was unable to convince myself that there was anything sloppier going on here with trisection than with quadrisection or even bisection. I was at another dead end.

I should point out that my goal was to discover why the angle was not trisected, and in a purely Euclidean way. I didn't care that Wantzel had proven the solution was unattainable in 1837. And I didn't care because that story did not continue, "And so all the formerly trisected angles had to be thrown out and their trisectors hung their heads in shame." Euclid knew that what was in front of me was not trisected. I wanted to know how he knew. But I appeared to be too stupid to figure it out. So I punted and learned ...

You Are Alone with Your Own Being and the Reality of Things

I would leave out this part of the story. But that would be a disservice to other free-range mathematicians who find themselves in similar straights. I will make it short.

I asked for help with this problem of unyielding thirds. And I didn't get any. You can imagine why. I'm sure some math departments have a filter set up to send any off-campus email with the word "Euclid" or "trisection" into /dev/null. I could certainly write them one if they asked.

I tried wit. I tried humility. I even tried to trick people into answering. Nothing worked. So either I would solve this problem myself on my own terms or I wouldn't. I should also admit to ...

A Brief Moment of Insanity

At some point, after the thirdnesses of the construction appeared absolute and mental fatigue from banging my tiny head against threesies set in, I began to wonder if this angle wasn't actually trisected after all. Everyone from Euclid to Gregory Bateson has insisted that "every schoolboy knows" number is not magnitude. But my trisection was grounded in ratios of magnitude, which requires Euclid V, which almost no one, statistically speaking, understands. Had no one made the connection that magnitude and number, properly treated, were in fact the same thing? Had those people who did make the connection been as uninterested in the problem as I had been or perhaps had walked away because of Wantzel? (Newton in the first group, De Morgan in the second) Was I not possibly truly amazing? I mean ... anything's possible...

This insanity lasted only long enough for me to recall that trisection can be performed on a line by people who make it no further than Euclid I 34. And so my brief twilight dreams of a Charlie Rose interview and, possibly, an honorary doctorate from Phoenix University On-Line, quickly dissipated. I would have to find ...

My Own Euclidean Solution

I'm not actually as dim as I have been portraying myself. My degree in mathematics is from a perfectly respectable university and my grade-point average was still low honors in spite of taking a beating for a couple of semesters there in the middle where nothing you learn for one mathematic appears to have anything to do with the next.

But I am awfully slow.

This whole story takes place in a week. So around day five, in spite of my slowness, the Euclidean solution came to me. I'm going to require you to make another drawing in your head.

- Take your angle, draw your chord, and trisect as before.
- Select one of the lines of trisection as the radius and strike an arc across the angle.
- Create your new sub-chords on either side of the central, now sub-, chord
- With center as one end of central chord, radius of that central chord, draw a circle
- Circle meets other end of central chord but passes beyond outer chord.
- Thus, the original arc was not trisected. [QED]

Having reached a solution on my own terms was very satisfying. I had set my own, purely Euclidean, bar and had cleared it under my own power. But just as I turned away to leave this whole unasked-for experience in the past, one final question came to me ...

Where are the Points of Trisection?

This is not a Euclidean question. So I felt no compunction to stick with my straight stick and compass. Let's do another construction in our heads.

Go back to the bisected trisected triangle laying on its bisector, which is a right triangle now. If the angle to be trisected is A, the face of this triangle is A/2. And the face below the actual point of trisection is determined by A/6. The point itself is where the radius of A/6 lies above the cosine of A/2. So the location of the point of trisection, as a ratio to the half-chord, is

$$\frac{\sin(A/6)\cos(A/2)}{\sin(A/2)}$$

Here is a table of angles from 0.002 to 179.98 degrees by half-angles and the value of the above:

0.001	.3333333333244
0.01	.33333333244
2	0.333298
7	0.332896
15	0.331322
30	0.32527
45	0.315118
60	0.300767
75	0.28207
83	0.270252
88	0.262191
89.99	0.258836

You can see that at fifteen degrees, a trisected chord's margin of error is only 0.002011th of the chord itself. Clearly, on ordinary notebook paper, trisection is possible for small angles up to fifteen degrees. With a Euclidean stick in the sand and a reasonably-sized construction, you might go up to twenty degrees.

You will recall that I promised to reveal the method by which all angles can be perfectly trisected with ruler and compass. And most of you can already feel this coming and are wincing:

> **Corollary to the Constructed Relation of the Point of Trisection to the Chord.**
> Given the results of small angles above, it is clear that, given a large enough pencil lead and a small enough angle, all angles are trisected by trisecting the chord, because the margin of error falls into the width of the pencil mark.

I trust no one will have the temerity to argue with something so self-evidently true. I would be the first to admit that such angles and their trisections fall under Heisenberg's Uncertainty Principle in that one can know that they are trisected or one can know what the angle is but one can no longer tell one angle from another.

Both his principle and my Euclidean construction fall under the overarching principle that "If the things you study are small enough, you no longer know what you are talking about." And this reminds me of my promised later resonance for the title of this essay. We can now ask, as earlier geometers and theologians once asked,

> How many trisected angles can dance on the head of a pin?

The answer to which, I leave as an exercise for the reader. That last phrase in itself makes all this seem much more mathematical, doesn't it?

You have probably also forgotten my final promise which was that there was actually a point to this essay. And knowing me as well as you now do, you would probably not be surprised if there was no further point at all. Surprisingly, to me certainly, there is a point. And that point is ...

What Euclid Meant by the Term "Angle"

I did almost forget the quadrature of the circle. That's not the point of the essay. But I'll throw it in here for what it's worth. The problem is basically, What is the line that is equal to the circumference of a given circle. If you could move arc to line, and if you could move the above trigonometric ratio on the open interval of zero to pi over two to a proportional line, you could use the chord to get the point of trisections by Euclidean means. I don't expect this to happen any time soon.

Let's go back to Euclid's angles. In Definitions I, he says:

> 8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
> 9. And when the lines containing the angle are straight, the angle is called rectilineal.

And this is all we get. This is all we have from the definitions when we want to understand the bi- or trisection of an angle. But Euclid knows there is more to angles than this. He doesn't want to say it. But he gives himself away in VI 33:

> In equal circles, angles on center or on circumference have the same ratio as the arcs subtended. This is true also for sectors.

Note that he says nothing here of chords. Euclid knows that the chords do not stand in any kind of ratio, as he understands ratio, to anything. They should. But they don't.

There have been two metaphysical cataclysms resulting from mathematics which have affected world thought. The most recent was non-Euclidean geometry which cause people to fundamentally doubt what was true in mathematics and science and elsewhere. I think this was silly. But I won't go into that now.

The earlier catastrophe of thought was the incommensurable, or real, number. This was first encountered as the diagonal of any square, which was never in a rational proportion to its sides. This fact revealed to the Pythagoreans that the natural numbers were not the voice and substance of the Creator. And from that time forward, they shied away from the incommensurables as best they could.

But Euclid knew that the chord of an arc did not define an angle. It was not is a calculable relation to arc or diameter. It was, to him, the face of a regular n-gon, inscribed in the circle. I'm not claiming to be channeling Euclid's mind here. But I'm pretty sure I can say the following:

> **For Euclid, the measure or magnitude of an angle, that which must be trisected, was the arc of any circle with the angle's apex as center. But he couldn't bring himself to say so, possibly because no ratio including the chord could be established. Possibly because no single chord or arc could be chosen as definitive. They all worked equally well.**

That, I suppose, and for what it's worth, is the point of this essay. The whole trisection experience was rather unpleasant. Though, it was nice to find a Euclidean disproof of trisection. And the trigonometry was fun to play with. But the only thing that seems to linger from this inquiry, is this idea of Euclid and the "archangel", the metaphysical place in his thought of the arc of an angle.

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