Remarks upon the Form of Number

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0. Hints

> If we say that this area of our subject is extraordinarily difficult, that is not true in the sense that we are, say, talking about things which are extraordinarily complicated or difficult to imagine, but only in the sense that it is extraordinarily difficult to negotiate the countless pitfalls language puts in our path here. ---Ludwig Wittgenstein
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> And so the development of [our] mathematic consists of a long, secret and finally victorious battle against the notion of magnitude. ---Oswald Spengler
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> The nature of infinity is this:
> That every thing has its own Vortex,
> And when once a traveller thro' Eternity has pass'd that Vortex,
> He perceives it roll backward behind His path,
> Into a globe itself infolding like a sun,
> Or like a moon,
> Or like a universe of starry majesty,
> While he keeps onwards in his wondrous journey on the earth,
> Or like a human form, a friend with whom he liv'd benevolent.
> ---William Blake

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1. The Form As We Find It

A criticism of mathematics must distinguish between the necessary, and essential, arbitrary nature of formal expression and the unnecessary ambiguity that is the cause of contradiction and that is eliminated by clarity.

This work is such a criticism of mathematics. A superficial reading of it will give it the appearance of a critique of Cantor's Diagonal Proof and its consequences. But Cantor's proof was a destructive gesture and these remarks are a refutation of its underlying presumption.

The first part of these remarks may appear to be an historical catalogue of mathematical ideas so accepted and so fundamental as to be beneath our notice. But the essence of this critique is not historical. And these basic ideas, upon which mathematical analysis appears to rest, make up the picture that these remarks would call into question.

The difficulty in doing mathematics, that is to say, in understanding it, is to remove from thought all but the very little required in each instance. This critique attempts to remove what is unnecessary from the present picture of the foundation of number. An acceptance of this criticism would appear, at first, to undo some parts of accepted analysis and related mathematics. But the effect of these remarks will be seen to clarify and not to destroy. No real mathematical object will be lost and a great many ideas will stand out more clearly against the moving background of mathematical thought.

The idea of number is an inclusive one. Historically, any object that behaved, under the operations of number, as accepted numbers did, found itself gathered into the fold of number. In the beginning God created the natural numbers or, at any rate, a world very suggestive of them--so wiet hat Kronecker recht gehabt. And in the beginning of what we consider mathematics, number was the naming of extension. The undefinable primitive was the unit. Given a choice of unit, one extended object could be expressed in terms of another, or so one hoped. This was ratio. And those ratios that could not be expressed were alogos, beyond thought. These relations beyond the scope of thought have been called incommensurable. Thanks to Rome, they are embraced by the word irrational.

Two remarks are in order. That which was alogos was not a number; it was the absence of a particular aspect of the very relation that defined number. And while the alogos was beyond thought, it was not--for the Hellene--an object in the infinite distance. The alogos was that for which there was no thought. It was a hole in the Greek mind.

The ratios of magnitude or the comparable lengths of lines were kept separate from the counting numbers as being distinct forms. This was the practice of some mathematicians as late as the fifteenth century. But the inclusive nature of mathematical thought was at work from the beginning.

The Pythagorean mathematicians did not distinguish between number and geometrical points. They appear to have thought of numbers as small spheres of negligible extension. Euclid's axioms removed all extension from points, making them dimensionless intersections of two lines having only length. Soon after Euclid, we come upon Aristotle, digging what is to be one of the most enduring pitfalls of language in mathematics:

> Aristotle discusses the fundamental problem of how points and lines can be related. A point, he says, is indivisible and has position. But then no accumulation of points, however far it may be carried , can give us anything divisible, whereas of course a line is a divisible magnitude. Hence points cannot make up anything continuous like a line, for a point cannot be contiguous with a point. A point, he says, is like the now in time; now is indivisible and not a part of time. A point may be an extremity, beginning, or divider of a line but it is not a part of it or of magnitude. He also argues that a point has no length and so if a line were composed of points, it would have no length.
> Mathematical Thought from Ancient to Modern Times, p.52
> Morris Kline, Oxford Press. 1972

To say that a point may be the extremity of a line is to say the extremity has position. But this position is an ideal one in Aristotle's mind and without dimension. What is left if we remove the point at the extremity? Does the line no longer terminate at the former position? One can see that Aristotle is combining the grammar of the dimensionless point with the grammar of magnitude. The temptation to combine these grammars come from Euclidean considerations. If the intersection of two lines exist, their point of intersection, as an object, must exist. If we construct a line with our Euclidean ruler, we feel we have Platonically constructed all its points. Yet for Aristotle, the points are equally the effect of our mathematical manipulations, arising here or there as necessary but otherwise absent.

To say that a point is a divider of a line is to say the removal of a point leaves two magnitudes that, together, make up the original magnitude. And so no point may contribute to the magnitude of a line. Again this arises grammatically from Euclidean construction. These divided magnitudes, if commensurable, and the operations upon them would have been distinguishable from the counting numbers or natural numbers and their operations only in a formal sense, that is to say, axiomatically. An axiom is the insistence upon a picture we choose to maintain.

And to say that a point is no part of magnitude is to say magnitude is related to the unit and not to the sum of points. We see here in Aristotle, a point with qualities familiar to us. For all its presence, its denseness, in the line, the point has no magnitude. The point is all but nonexistent; it is as far from magnitude as can be imagined.

The predicament of the incommensurable magnitude was that the laws of magnitude operating on magnitude produced an object outside magnitude. As mathematics progressed, the standpoint of mathematicians towards incommensurables as number depended upon motive. Those like Heron and Archimedes, for whom mathematics could be a means, accepted the incommensurable and approximated it for their ends. Those like Diophantus, developing mathematics for itself, rejected the alogos and would even alter an equations containing one so as to produce a different answer--one that would lie within the bounds of thought.

The Hindu and Arab cultures, from which we have borrowed artifacts for our own mathematics, operated upon rational and irrational alike. These thinkers seem to have been unconcerned with the difficulties inherent in the form of the incommensurable. Curiously, the Hindu accepted negative number for debts and the like while the Arab rejected the negative number altogether, though the Hindu scholars lived in their midst.

A distinction arises here that is central to these remarks. I borrow two concepts form the work of Gottlob Frege and delineate them for my own use. These are the concepts of sense and reference. Sense is the consistency of an idea within its mathematic. Sense is the meaning arising from the idea's formal context. Sense often reduces to the question: is this object constructible. Construction here is not merely a question of finite steps but of clarity in the development of the idea.

Reference is the connection of a mathematical idea to an object in the only world. Reference is not a mapping of mathematical ideas upon the physical world; that would be nonsense. Reference is the connection of a mathematical idea to the principled universe of discourse, of which mathematics is only a subset.

Sense is internal coherence; reference is the object's place in mankind's understanding.

The concepts of sense and reference can be applied as a type of Occam's razor to the objects of mathematics. Consider the way that linear algebra studies linear transformations and their corresponding matrices. It is clear that we know the transformation by nothing other than its matrix or some series of expressions equivalent to that matrix. The matrix is the rhetorical invariant of any expression through which we know the transformation. This is to say that the matrix is the only knowable object of the transformation.

In terms of this example, the matrix has the sense arising from the formal calculations of algebra. And in this particular context, the transformation of a linear space is the reference of these expressions. But there can be ambiguities of reference. Ideas represented by matrices may refer to knots or tensors or the incidences of a directed graph. Or an idea may have no referential aspect. Objects having only a sense are the tautological or contradictory forms of any abstract formal context.

The struggle of mathematical cultures to find a consistent standpoint of the incommensurable magnitude can be seen in terms of sense and reference. The reference of the incommensurable was the ratio of particular magnitudes or the consequence of a particular Diophantine equation. But the incommensurable was not expressible in terms of any other mathematical objects. And so its sense became a contradiction. Only where the reference of the alogos seemed to be a clear mapping onto the understanding of material world, as in the area of a piece of land or the diagonal of a visible square, did the reference outweigh the contradictory nature of its formal sense. Given the assurance of such a mapping, itself only a particular standpoint of the mind, sense was put in abeyance and the operations of number were allowed to operate upon this hole in the Greek mind.

In the Dark Ages, mathematics became bound up in a catholicism of appeal to the authority of Aristotle. And so, our survey of the form of number in the mathematical mind can well pass on to the sixteenth and seventeenth centuries.

In the sixteenth century, while zero had been accepted as number, negative numbers had yet to be brought fully into the fold. Michael Stifel considered them absurd. Francois Vieta refused to have anything to do with them. But the conformity of negative numbers in the formal sense was overcoming the question of reference in mathematical thought. Simon Stevin accepted negatives as roots and as coefficients; for him, they were as well-behaved as the positive numbers. Thomas Harriot accepted negatives in some contexts but not as roots. Rene Descartes took a similar stance, one reminiscent of Diophantus. For Descartes, a negative root representing less than nothing was not acceptable. But given an equation with negative roots, he would derive another with positive roots and so accepted negative numbers as essential to the process of his mathematic.

In the seventeenth century, negative numbers were still in dispute. Blaise Pascal held them to be nonsense. Antoine Arnauld reasoned that the equation:

> 1/-1 = -1/1

led to a contradiction and rejected negative number. John Wallis accepted them. And Gottfried Leibniz took the middle ground. While he agreed with certain criticisms of negative number, Leibniz argued that one could include them in a calculus because they adhered to the form of number.

During these centuries, the mathematical mind was still working on the contradictions inherent in the form of the alogos In his Arithmetica Integra of 1544, Michael Stifel wrote:

> Since, in proving geometrical figures, when rational numbers fail us irrational numbers take their place and prove exactly those things which rational numbers could not prove . . . we are moved and compelled to assert that they truly are numbers, compelled that is, by the results which follow from their use--results which we perceive to be real, certain, and constant. On the other hand, other considerations compell us to deny that irrational numbers are numbers at all. To wit, when we seek to submit them to [decimal representation] . . . we find that they flee away perpetually, so that not one of them can be apprehended precisely in itself .... Now that cannot be called a true number which is of such nature that it lacks precision .... Therefore, just as an infinite number is not a number, so an irrational number is not a true number, but lies hidden in a kind of cloud of infinity.
> Mathematical Thought from Ancient to Modern Times, p. 251.
> Morris Kline, Oxford Press, 1972.

Simon Stevin accepted the incommensurables under the form of number and approximated their value with rational expressions. Pascal restricted the incommensurable to its Greek form of denoting geometrical magnitude. Descartes hedged his bet by stating that irrationals are abstract numbers and represent continuous magnitudes. Abstract as compared to what? For Descartes the answer would be: Abstract as compared to the number that clearly had reference.

The form of the alogos is bound up with the form of the infinite. These same mathematicians were struggling to make sense of infinite form as it applied to number. Algebraically, Wallis reasoned that since a number n divided by zero equalled infinity, n divided by a negative number was greater than infinity. This was applying the grammar of arithmetic's monotonicity to the grammar of infinity. Wallis was also developing the idea of convergence, not in the convergences of sequences, but for the continued fractions that represented irrationals.

In geometry, the infinite was leading towards the results of Newton. Cavalieri was upholding the

Aristotelian picture of lines made of points and extending this to model solids built from planes in his work on volumes. He allowed for an infinite number of points or planes in these models of length and volume.

Two images were being developed in these mathematics. One was the image of an infinite number of succesively smaller values or increments that as a whole represented a number. The other was the image of length, area, or volume comprised of an infinite number of elements of the next smaller dimension. In the Calculus of Newton and in that of Leibniz, we find these pictures used to uphold two distinct and even more fundamental concepts.

For Leibniz, the Calculus was a transformation of x into dx, and dx into x. For him, the dx are infinitely small in the sense of absolutely small. The are the atoms of Democritus (the ones Blake warned us about). One can see in his writings that he presumes the existence of these particles:

> In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included.
> Mathematical Thought from Ancient to Modern Times, p. 385.
> Morris Kline, Oxford Press, 1972.

Here, Leibniz is a Pythagorean: for him, a number and the dx that lead to number are objects, entities. He has fallen into a confusion of language that Plato tried to clarify in the Sophist, namely, that any object that can be expressed must necessarily have existence for the non-existent cannot anywhere exist, not even in language. This is, of course, a central problem of language. And geometrically expressed, this error is the belief that any real road must have a real terminus, to use Leibniz's words.

For Newton, the Calculus was the application of law, rather than a construction upon the indivisible. He writes of integration:

> I considered mathematical quantities in this place not as consisting of very small parts, but as described by a continuous motion.
> Mathematical Thought from Ancient to Modern Times, p. 339.
> Morris Kline, Oxford Press, 1972.

And he writes elsewhere:

> Fluxions are, a near as we please, as the increments of fluents generated in times, equal and as small as possible, and to speak accurately, they are in the prime ratio of nascent increments.
> Mathematical Thought from Ancient to Modern Times, p. 363.
> Morris Kline, Oxford Press, 1972.

The student of Calculus will recognize the prime ratio of nascent increments:

> dy/dx = nx^(n-1).

Newton is not requiring that the road down through smaller and smaller increments have a terminus. He requires only that the road be real and, as real, then subject to law.

In the Principia, Newton expresses the idea that the quantity need not be an object:

> Ultimate ratios in which quantities vanish are not, strictly speaking, ratios of ultimate quantities, but limits to which the ratios of these quantities, decreasing without limit, approach, and which, though they can come nearer than any given difference whatever, they can neither pass over not attain before the quantities have diminished indefinitely.
> Mathematical Thought from Ancient to Modern Times, p. 365.
> Morris Kline, Oxford Press, 1972. Geometrically, for Newton, the substance behind the

> idea of the infinitesimal is that only arbitrary closeness to a point matters, not the point itself. And so for him, integration is not the summation of infinitesimals. It is simply the application of the inverse of the derivative law.

Newton's use of geometry throughout his work suggests that if he could have put the Calculus upon a solid constructive basis in geometry, he would have done so. But this basis would have required a resolution of the form of the alogos. So, rather than put his Calculus upon a pseudo-constructive basis as Leibniz had attempted to do, Newton places it upon the strongest basis possible--that of adherence to law. He seems to assert that a picture of the law cannot be an explanation of the law, nor is such a picture needed to support the law's operation.

It is the claim of Oswald Spengler that for our culture, for our mathematic, function is number:

> The symbol of the West is an idea of which no other Culture gives even a hint, the idea of Function. Function is anything rather than an extension of, it is complete emancipation from, any pre-existent idea of number. Not only Euclidean geometry ... but also the Archimedean arithmetic, ceased to have any value for the really significant mathematic of Western Europe.
> The Decline of the West, p. 55.
> Oswald Spengler, Knopf, 1962.

Function comes to power in the eighteenth century. And it is put upon the throne by Leonhard Euler who dominates the mathematic of his century as Johann Sebastian Bach dominated the form of music in the same years. There is no comparable shaper of mathematics after Euler until the dark Beethoven of Gauss.

In Euler's time, irrational and imaginary numbers are widely used and accepted. The problems inherent in their form have not been solved--indeed they are scarcely addressed. For the form of number is being subjugated by the form of function. And these functions are showing the consistent usefulness of number's problematic forms.

The comparison of Euler to Bach is apt. As Bach was forever tinkering with the formal structure of music, Euler's work is replete with the manipulations of function. He developed the ideas of infinite series and convergence. Into these series, he brought the polynomial functions, the logarithmic and trigonometric functions. And these forms of number, which in themselves eluded conformity to reference, operated harmoniously in these Eulerian fugues.

Euler played freely in the fields of infinity. For him, infinity was a number, the infinite sum of all the natural counting numbers. He further delineated orders of infinity. An number n divided by zero was of the first order; n divided by dx^2 was second order infinity, and so forth. He established the transcendental functions as infinite series of algebraic functions.

With Euler, mathematics takes on a more formal aspect. In certain cases, as with infinity, he allows the internal consistency to support the idea in the absence of reference. While he is moving mathematics away from reference, his writings show his thought still reaches towards his need for reference. In 1755, he writes of infinitesimals:

> There is no doubt that every quantity can be diminished to such an extent that it vanishes completely completely and disappears. But an infinitely small quantity is nothing other than a vanishing quantity and therefore the thing itself equals zero. It is in harmony with that definition of infinitely small things, by which the things are said to be less than any assignable quantity; it would certainly have to be nothing; for unless it is equal to zero, an equal quantity can be assigned to it, which is contrary to the hypothesis.
> Mathematical Thought from Ancient to Modern Times, p. 429.
> Morris Kline, Oxford Press, 1972.

And in 1768, concerning complex numbers:

> Because all conceivable numbers are either greater than zero, or less than zero or equal to zero, then it is clear that the square roots of negative numbers cannot be included among the possible numbers .... We must say that these are impossible numbers.
> Mathematical Thought from Ancient to Modern Times, p. 594.
> Morris Kline, Oxford Press, 1972.

In the first quote above, Euler is trying to establish the clarity in the development of the limit idea. The infinitely small quantities clearly have a sense within the Calculus. But for Euler, they are without a clear development, or logical basis. Euler is hindered here by the linguistic pitfall of existence into which Leibniz fell. Where Newton makes do with a process delineated by law, Euler searches for the existent thing. In the last quote, the sense of complex numbers causes the reference to be contradictory. Yet Euler does not throw out these impossible roots. Because they conform to the laws of number, he keeps them as number.

With the nineteenth century, the canopy of number had beneath it most of the objects we recognize today as number. Skirmishing over reference and logical consequences continued. But number included the natural, negative, rational, irrational, and complex. Grassmann and Hamilton would be along shortly with more numbers. The landscape is familiar and all of these forms are at work in the development of function.

In the early part of the nineteenth century, some mathematicians were feeling a need for a clear foundation for the forms of number underlying the real number system, the complex plane, and the structural images representing these forms in their totalities. Much of the problem in building a basis for these ideas was inherited from the hole in the Greek mind. Decimal representation had been in use since the late sixteenth century. And now the irrational in that representation had been brought into the fold although it failed to consummate itself in that representation. For the form the alogos took in decimal representation was an infinite one. A number with infinite form was not directly representable and was not subject to the same analysis as the rational numbers. The irrationals were outside the rigor of the nineteenth century.

Two processes upheld this infinite representation. The first was the division algorithm that converted rationals into decimals. Many of these rationals have a nonterminating decimal form that is an infinite repetition of a pattern of digits. This image of an infinitely long decimal form with a pattern evoked the image of an infinitely long decimal without a pattern. One might remark that the non-termination of repeating decimals has nothing to do with infinite form but is solely a consequence of the operations of number, division in this case. The infinite form of irrationals was also supported by approximations of irrationals, as in the expression of continued fractions and in proofs such as Fermat's methods of infinite descent.

While the tools for clarifying the irrationals were being forged, some mathematicians showed a willingness to more or less side-step the whole question of explanation. In his Cours d'Analyse de l'Ecole Polytechnique in 1821, Cauchy defined the irrational as the limit of distinct fractions approximating the irrational more and more closely. This was at least an impredicative definition, if not something worse, and Cauchy dropped it from later editions.

Other mathematicians were having difficulties with reference. In 1831, DeMorgan wrote, in his work, On the Study and Difficulties of Mathematics:

> The imaginary expression [the square root of negative a] and the negative expression [negative b] have this resemblance, that either of them occurring as the solution of a problem indicates some inconsistency or absurdity. As far as real meaning is concerned, both are equally imaginary, since [zero minus a] is as inconceivable as [the square root of negative a].
> Mathematical Thought from Ancient to Modern Times, p. 593.
> Morris Kline, Oxford Press, 1972.

De Morgan, along with others, began in the middle of the nineteenth century to abstract algebra away from reference in order to work solely with the formal sense.

One of the best tools for clarifying the irrationals was forged by Dedekind in 1872. His picture of the reals is roughly as follows--

The consistency of the rationals, Q, is assumed. Q is shown to be dense in the reals, R. This is to say, topologically, that every r in R is either in Q or is a limit point of Q. Q is shown to be discontinuous; else Q and R would be the same set. Then Dedekind defines a cut of the reals to be two classes (A1, A2) which define a real number r. For every possible r, all elements in A1 are less than all elements of A2. It is shown that there is either a least element of A2, a greatest element of A1, or neither. In the first two cases, r is a rational number, an element of Q. In the third case, r is irrational. Philosophically, Dedekind attends the same school as Leibniz. For Dedekind, the cut was not r. Each r had a "real" existence of its own and caused the cut.

It could be argued that Plato attended this school and that Socrates did not.

This claim for existence was criticised by Heinrich Weber, whose obscurity attests to the interest mathematicians take in the philosophical underpinnings of their work. Georg Cantor criticised the whole of Dedekind's construction as hingeing upon cuts that had never, until Dedekind, arisen in any mathematic.

Cantor was hard at work forging his own tools. The line, its image unchanged since Aristotle, was finding its place as the structural image of the set of real numbers. In 1872, Cantor asserted for the line what Newton had asserted for number in 1707, namely that every number, however it arose, had its place within the structure of the line.

The following year, 1873, Cantor published his famous proof that the cardinality of the set of real numbers is greater than the cardinality of the rational numbers. This was Cantor's Diagonal Proof, the CDP, and is the pivot and essential cause of these remarks. And so its exposition will have a chapter of its own. The CDP was a vindication for the largely accepted picture of the real numbers. Cantor transfigured the imperfect irrational, giving it infinitely more presence than the rational.

The CDP led to Cantor's theory of the transfinite numbers. The natural numbers, those we count with and that are subject to enumeration, were assigned the cardinality aleph[0]. The set of all subsets of the counting numbers, now held by most mathematicians to be the set of the reals, was assigned aleph[1]. The set of the transfinites as a whole were the sets aleph[i] where i was the set (0, 1, 2, 3, ...) and aleph[i] was the set of all subsets of aleph[(i-1)]. This was a picture of infinite infinities, beginning with the countable ones and passing beyond human apprehension into the uncountable ones. This infinite set theory soon found a place in fields such as topology, analysis, and measure theory.

But is was not without critics. Kronecker, of course, objected. For Kronecker held that the irrationals simply did not exist. In the dilemma of number in modern times, most mathematicians were impaled either upon the irrational horn or upon both horns at once. Only Kronecker is left dangling from the rational horn. Other objectors to Cantor's picture of the transfinites were Felix Klein, the inheritor of Riemann's chair of mathematics, Henri Poincare, and Hermann Weyl. Transfinite set theory gave rise to the Axiom of Choice and its equivalents. This axiom was criticised by Emile Borel, Jaques Hadamard, and Henri Lesbesgue. Lesbesgue rejected the proof of the existence of transfinites and apparently held to this standpoint in the face of Borel's and Hadamard's insistence that his stance was equivalent to the rejection of the reals.

None of these mathematics were a final resolution of the difficulties inherent in the alogos. Cantor went on to construct his own picture of the reals based upon sequences of rationals in 1883. But this attempt solved little or nothing. The mathematics of set theory came to resemble the mathematics of the Calculus after Newton. It was leading to new and fruitful mathematics and these child-fruits seemed to justify the vague foundation of the parental ideas. Yet no foundation that clearly resolved the problems of number had been found for either parent and all such problems are hereditary.

In the early nineteenth century, Nils Abel had seen the need for a rigorous foundation for real analysis and the seeds of his desire had continued to grow in the mathematical mind. Some mathematicians, however, were showing a desire to avoid many of the difficulties by throwing off reference completely. As early as 1887, Emil Du Bois-Raymond asserted that mathematics could be reduced to a chess game without reference. This was to say that mathematics was solely a formal language, its sense self-contained. Soon after the turn of the twentieth century, this formal approach led to two schools of thought that sought to build foundations beneath the mathematics of the real numbers.

Alfred North Whitehead and Bertrand Russell, building upon the work of Gottlob Frege and Guiseppe Peano, attempted to build the fundamental forms of number using the forms of formal logic as primitives. This became known as the logical school of mathematical foundations. Whitehead's and Russell's work was expressed in the three volumes of the Principia Mathematica, written almost entirely in formal symbolic logic.

The good ship Principia, which remained manned even after Russell's death, actually ran aground in the early part of his career upon the reef of the alogos. Russell was working on the idea of sets that were not an element of themselves. These sets arose from the form of the irrational, each irrational being a member of a set made up of all possible combinations of natural numbers. A definition that defines an object by means of a class of objects having the defined as a member is called an impredicative definition. Russell's study led to the set of all sets that do not have themselves all elements. This set, unfortunately, was and was not such a set. As such it tore a long, icy hole in the Principia's hull.

As a patch, Russell introduced his Theory of Types. The first type was the set of numbers. The second was the set of the properties of numbers. The third was the set of the properties of those properties. And so on. This was his attempt to place problematic sets in a separate type than that from which the problems arose. In form the Theory of Types is indistinguishable from the theory of the transfinites. Both have the identical grammar. In the case of Russell's theory, each type is number; the theory as a whole is a renaming of the naturals and their consequences. Occam warned us against this sort of thing. And nothing was clarified or brought into understanding by means of these Types.

This critique will return later to both the idea of impredicative definitions and the underlying form of theories that rely upon them.

The second serious attempt at foundations followed more closely the lead of Du Bois-Raymond. David Hilbert attacked the problem of number by first laying down a groundwork of axioms. These conformed to the established form of number and the operations of number. But rather than claiming any basis, the axioms were openly held to be arbitrary, in the sense that the rules of any game are arbitrary. While he worked to describe mathematics as clearly as possible through axioms, Hilbert's formalism had the additional goal of proving that these axioms led to no contradictions. His methodology in this attempt he called metamathematics. Unlike Russell and Whitehead, Hilbert chose not logic but mathematic as the material of his foundation. His metamathematic eschewed what Hilbert could not clearly establish as adhering to the form of number: the axiom of choice, transfinite induction, and proof of existence by contradiction.

Hilbert's program ran aground upon those selfsame impredicative shoals that gutted the Principia. While metamathematics took a different form than the Theory of Types, it was again a false claim to a superior standpoint that was at bottom merely a renaming of objects. Hilbert was not manipulating mathematics with some higher set of meta-tools. He was practicing mathematics upon mathematics. When we truly have a higher standpoint, our entire understanding undergoes a transformation. A real metamathematical standpoint would amount to more than a justification of our limited understanding and its artifacts.

Goedel's famous proof of 1931 established that the set of all consequences of a denumerable set of axioms could not establish the truth or falsehood of all statements arising from those axioms. And so Hilbert's program came to an end.

All attempts to bring civilization to the alogos failed. When it was clear that the hole in the Greek mind was not only in our mind as well but had furthermore plenty of room for all of our attempts to fill it, the schools of mathematical foundations put "Here there be Tygers" signs all around the little hole and walked away.

The current form of number is little changed from the days of Hilbert and Whitehead. The real numbers include the natural numbers, the rationals, and the irrationals. All of these have shown their conformity to the ancient laws of number. The structural image of the reals is the real number line. And Cantor's axiom of 1872, that each number has its place therein has been made absolete by the Axiom of Completeness. This states that there are no gaps in the real line and thus no number can be produced that is not already represented there.

There are two more aspects of the form of the real numbers that bear upon this critique. The first is the denseness of the rational numbers within the reals. This property, which can be derived from the axioms of number, can be described by saying that between any two real numbers, a rational number can be constructed that is greater than the lesser of the two and less than the greater. This of course implies an infinite number of rationals between any two reals.

The second property is derived from the denseness of the rationals. It is the property that any open set of real numbers is made up of a countable number of open sets. This is to say that although there are uncountably many reals in any open set, it is yet made up of a countable number of sets.

And with this we conclude our description of the form of number as we find it. I am by no means oblivious to the failings of histories that rely upon secondary sources as this one has. But this critique is not historical. And I emphasize this by making the historical aspect a bit below the expectations of professional historians. I apologize if I have unwittingly passed along historical errors made by others or introduced any of my own; I take the responsibility for both.

History is neither more nor less arbitrary and creative than mathematics. I hope that mathematicians may be led to consider historical works such as I have used here. An understanding of mathematics in the full context of the only world cannot develop until mathematicians participate in such study. And the claim of this critique is that such an understanding is the source and substance of mathematics.

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2. Consistency and the Grammar of Infinity

A criticism of mathematics must address reference. It must be the claim for a particular standpoint regarding reference. The necessity of reference cannot be a connectedness solely to physical objects or their models. Reference is the adherence to principle or law, what can be called a moral necessity.

Sense is not concerned with the truth of a proposition. Sense has to do with tautological relationships among derived ideas and conventions among primitive ideas. The primitive ideas of mathematics were for ages understood to be derived from absolute truths drawn directly from our intuitions of the only world. But we now better understand these primitives to be free choices of thought and refer to them as essentially arbitrary axioms. Axioms may be thought of as the generative propositions of a mathematic.

Any mathematician may easily convince himself that mathematics based upon axioms need have nothing to do with truth or reference. The axioms themselves are the basis of necessity in such mathematics and need lead to no understanding outside of the expression of the particular mathematic with its internal consequences. This is to say that is need not effect us.

The generative propositions of mathematics have a psychological basis. This basis may also be understood to be artistic. The Axiom of Completeness, for instance, states that all possible infinitely long combinations of the ten digits exist as numbers. But whether we solve our need for completeness this way or in another is a psychological or artistic question. This is true of all fundamental ideas, for as Jose Ortega y Gasset pointed out it was at some point our decision to view all objects as separate from ourselves (our self) rather than as at one with it.

These generative propositions may be understood as the choosing of a picture. And it is with the choosing of the axioms and only at this point that choosing a picture is appropriate. The choices may be motivated by the creation of an entirely new picture, the alteration of an old one to bring it in line with better understanding, of the need to bring an orphaned part into the accepted whole.

But the subjugation of mathematics to the needs of psychology extends only to the generative propositions. Complete artistic freedom may be applied to the creation of a picture but not to its consequences. Once we settle the generative basis of axioms we have a mathematic. What we do within the mathematic is then a mathematical or formal question. And that doing must adhere to the principles of mathematics. Each mathematic is subject to the need for internal consistency. Yet each object in each mathematic must adhere to our perception of consistency of all mathematics taken as a whole. An object subject to and conforming to law must be consistently subject to those same principles anywhere that object arises in the context of a mathematic.

Ludwig Wittgenstein expressed the necessity of adherence to principle in this way:

> The whole approach that if a proposition is valid for one region of mathematics it need not necessarily be valid for another region as well, is quite out of place in mathematics, completely contrary to its essence. Although many authors hold just this approach in order to be particularly subtle and to combat criticism.
> Philosohical Grammar, p. 458.
> Ludwig Wittgenstein, Univ. of Calif. 1978.
> (retranslated by the author)

Very few mathematicians would consider the objects of elementary analysis and the real number system to be inconsistent within analysis itself or the larger whole of mathematics. Yet it is within elementary analysis, or rather in introductory calculus, that a student is introduced to two objects bound up in the fundamental form of the alogos. They are the decimal representation of the irrational numbers and the cardinality of the natural numbers.

It is the claim of this critique that an object that does not adhere consistently to the laws of mathematics is not yet clearly expressed as an object in mathematics. In this section the two objects just mentioned will be stripped of their inconsistent rhetoric. We will show what form they must take when held to the necessity of principle. This clarifies the grammar of these objects. Further, we will see the effect of this clarified grammar upon the current rhetoric of the irrational number and of the cardinality of the natural numbers.

A warning is necessary here. The basis of reliance upon axioms conceals but does not replace the underlying picture of the real numbers that is the actual basis of mathematics. And when we appeal to those axioms, we are appealing to that picture of number and the authority of those that defined it. This section runs counter to this accepted picture of number. But these ideas are not a suggested replacement of that picture. The purpose here is to bring to light the false belief that must be cast out of the mathematical mind. The pivotal idea of this critique will then be seen as fully within the principled expression of mathematics.

Both the decimal representations of the irrational number and the cardinality of the natural numbers are expressions of infinity. Infinity is an aspect of their forms. And the idea of infinity can be clarified by the concept of the zero distance. Let us call the absolute values of all numbers that have been actually, historically, expressed by mankind the set E. This set has the property of being arbitrarily large. But because it is a set of artifacts it is, at any given moment, finite. Then let us say that a real number is within the zero distance of the origin if and only if its absolute value is less than or equal to a member of set E. Let us then construct the structural image of the zero distance. Let the largest element in E be e. No matter how large e is and no matter what the scale of the unit upon the line, the ratio of the distance from e to the origin to the length of the line is zero.

The zero distance is the set-theoretical equivalent of the decimal point. It is at once our standpoint and the absolute limit of that standpoint. This zero distance is not a construction. Its value is unchanging and shows our absolute relation to the form of number. We see by the zero distance that no picture of continuation in a proof speaks to infinity. Such a picture shows only exclusion from the zero distance. It shows the futility of verification of irrationality, transcendence. or any other aspect of infinity by finite methods, even high-speed mechanical ones. No finite progress can reach beyond finity. A proof of infinity must speak to the form of infinity, as do proofs by induction. No proof addressing merely the finite beginning of infinity can establish the relation of an object to an aspect of infinity.

We turn now to the inconsistencies of the present grammar of irrational numbers. The first claim of the present grammar is that there is no smallest irrational number. In handling irrational numbers, we lose no generality by limiting our discussion to the continuum, C, the interval (0, 1) on the real number line. A smallest real number would be the first number larger than zero. A real number in C has an expression of zero followed by a decimal point, followed by an infinite number of digits. By definition a number is determined if all its digits are determined. The smallest number is clearly determined as: 0.000...001 where ... represents an infinite number of zeroes.

The exclusion of this kind of determination from the present grammar of irrational numbers is based upon its relation to the decimal point. It is permissible to construct a determinate number from the decimal point, building infinity towards the descending values. But it is not permissible, currently, to state the latter value first and then to push it away from the decimal point, although the two methods are clearly equivalent. To be fair, no mathematician would claim that we that we have failed to determine all of the digits in this smallest number. Rather, in the present grammar, indeterminate simply means we cannot say how small this number is. We cannot locate its 1 in numerical space, so to speak, from the standpoint of the decimal point.

We must clarify the basis of this objection. It cannot be a logical or an a priori objection, for the number above conforms absolutely to the form of the irrational number. And the objection cannot be that its magnitude is not determined. It is determined in that it is the smallest possible number representable given the concept of infinity and the digits 0-9. It is not determined only in that we cannot write all of its digits down; we cannot say how many zeroes there are. But this is true of all real numbers that are not elements of our set E. This is to say that we cannot express explicitly essentially all of number.

The objection boils down to this:

If we cannot write a number explicitly, either digit by digit, or by a symbol of explicit reference, such as e or pi, we cannot use it in a calculus. Objects that conform to the calculi or operations of number are number. Therefore if an object cannot be expressed explicitly, it is both a number and not a number.

Such an objection is empirical , perhaps even utilitarian. Objections against non-Euclidean geometry sprang from the same basis. In making such an objection one reverts to a Hellenistic standpoint in the midst of our Faustian mathematic. I borrow from Spengler the concept of the Faustian or Western European-American culture that arose at the end of what we call the Dark Ages. But a mathematic of infinity cannot have an empirical or utilitarian basis. The empirical is finite.

We see here a clarification of our relation to mathematics. The standpoint of our calculi is not one of omniscience; it is clearly restricted. The calculi of our mathematic are limited to the demonstrable which is to say that they are constructive. They handle only manageable quantities: either those of the zero distance or those clearly referenced. The majority of the objects that we claim are in conformance to the law of number are inaccessible to our calculi.

We also see that there are regions of mathematics about which we can make no verifiable assertions. Infinity is a potential property and not a demonstrable property. No demonstration would suffice to show the representation of pi to be infinite in form. It is either demonstrated to be within or without the zero distance. No showing can speak to infinity. Proofs of aspects of infinite form must rest upon some non-demonstrative basis. If we say that a number has an infinite number of decimal places, this assertion has a sense. But it can have no reference. The sense here is itself the irrational number; the number merely the sense.

It clarifies our grammar of the irrational number to understand our use of the word arbitrary, in the sense that a number is arbitrarily small or is arbitrarily smaller than another number. This is a restriction of reference to the rational numbers. It is a part of the rhetoric of infinite sequences and series and of continued fractions. And applying this rhetoric to an irrational number does not say anything about infinity. The grammar of arbitrary approach to a limit takes place entirely within the zero distance. The structural image of approximation is composed of rational numbers. And at no point does this approximation pass over into infinity. If it did it would be possible to create a structural image of the elements of C ordered in such a way that the list of terminating and repeating decimals would pass over and continue into non-terminating and unpatterned ones. But the two are mutually exclusive.

The second object that is treated inconsistently in mathematics is the cardinality of the natural numbers. Its first inconsistency is in its symbolic expression. In mathematics, we do not constantly remind ourselves of the distinction between the ordinal 9 and the cardinality 9. Indeed, it would be hard to conceive of a usage where the distinction would arise. Yet the rhetoric of mathematics is constantly separating the ordinal N and its cardinality |N|.

During the first half of the twentieth century, it was not uncommon in textbooks to see the symbol of a sideways eight, equivalent to |N|, used with subscripts to distinguish between the cardinality of N and that of other infinities. The same symbols occur in calculus texts belonging to students who are quickly told that (symbolism aside) infinity is not a number. If we look outside pure bmathematics, however, we find physicists manipulating and cancelling infinities in their equations according to the common laws of number. The inclusive nature of number is at work upon infinity. Mathematicians, in this stagnant period, attempt to insulate themselves from this growth of usage, just as academics in the written arts bewail the introduction of already fluent words into the dictionary. But the current intellectual divisions of mathematician, physicist, astronomer, engineer arise from economic considerations. All are mathematicians and subject to the growth of mathematical thought.

It was Cantor's picture of the transfinites that introduced the current usage of N as a number. His theory arises from the concept of the set of all subsets, one consequence of which sank the Principia. Any finite set of n elements may be arranged into two to the nth power subsets, if we include the empty set. For example, if we have a set of two elements (a, b) we have two to the second power subsets: (a,b), (a), (b), ().

For Cantor, the cardinality of the natural numbers was a number. As the cardinality of a set of two elements was 2 the cardinality of the set of N elements was N. This cardinality he called aleph[0]. The cardinality of the set of all subsets of N, having 2 to the N elements, was aleph[1]. This is generally thought of as the set of real numbers and the first uncountable or non-denumerable set. The definition of cardinality being recursive, Cantor was led to define an aleph corresponding to each natural number, which gives us a messy, Augean stable of sets.

The claim that N is not to be used as a number can not be placed in a consistent context. The set of all subsets of N arises from the grammar of x to the y. And yet we are forbidden to use 2^N in mathematical calculi. There are two aspects of this rejection of 2^N. The first aspect is more obscure, more visceral so to say, than the earlier objection to the use of inexplicit numbers. It is that after the first few transfinites, no one has any idea as to what they are talking about when them speak of them. These are objects that rapidly depart from the possibility of reference of any kind. Beyond the bounds of reference the words "larger," "greater than," and "uncountably uncountable" are so purely formal as to refer to no object beyond their own words on the page or their sound issuing from a mouth. As such they are isolated concepts, sterile rather than generative propositions, and are neither understood nor cause any other object to be more clearly understood.

The second aspect of this rejection is that infinity is on the wrong side of the "Here there be Tygers" sign. The inconsistencies of the alogos do not arise only at the absolute limit of the set of all subsets. They arise immediately. We will now see examples of how these inconsistencies would quickly destroy our current rhetoric of infinity if we permitted the use of N as a number or accepted the explicit, yet indefinite, form of the irrational number I used above.

Consider such an explicit irrational. Let this irrational be i. It must be infinitesimally smaller than some other particular real number r, $i < r$. For if i were finitely smaller than r, i would be rational. By the density property of the rationals in the reals, there is a q in Q such that $i < q < r$. But $|r - q|$ is a finite amount and must always be greater $|r - i|$. This collapses r, q, and i into an identical number, $i = q = r$. An explicit expression of the irrational number destroys the basis of all "arbitrarily smaller" rhetoric. With the tail of an irrational explicit and the decimal-point end undefined, we find ourselves reduced to the rational numbers. The current rhetoric avoids this difficulty by speaking only of arbitrarily small quantities, which (as we have seen) is but another form of restricting ourselves to the rational numbers. Newton did what he did for a reason.

A "proof" like the above is another example of the error of thought Plato exposed in the Sophist. But all assertions concerning infinity contain this error. We are are unable to touch the infinite with our rhetoric and are left with the finite and rational. If we treat the alogos as an explicitly defined object, it fails to adhere to principle. If we are not explicit, the irrational is reduced to a cloud of "arbitrarily"s and a picture of an infinite number of digits beyond the decimal point.

Were we to treat the irrational as a definite object, the denseness of Q would disappear. For there is no number between 0.000...0001 and 0.000...002. And with the fall of density, fall all the myriad theorems depending from that picture. Among these is the theorem that every open set S of the real line is the union of a finite or countable set of pairwise, disjoint open intervals. Without the density of Q in R and given Cantor's Diagonal Proof, we would accept open sets made up of uncountably many open intervals. But our structural image of the real line is reliant upon the density of Q. This property plays the role of keeping any two explicit points apart. Without this property our zero-dimensional points would collapse the line unless, like Aristotle, we again made the structure of the line independent of the uncountable points within it.

The definite irrational would play havoc with other consequences of Cantor's theory as well. We know that the cardinality of the rationals is the same as the cardinality of the natural numbers, or

$$|Q| = |N|.$$

Assume, with Cantor,

$$|R| > |Q|.$$

Let each r in R be expressed in its entirety. No generality is lost if instead of all of Q we use its subset of infinitely repeating patterns of digits. We define the greatest lesser transformation for r in R--glt(r)--as the q that is less than r and may be made identical with r by changing the least decimal value of digits. This function is certainly onto and by assumption of R having the greater cardinality, thanks to Cantor, there are at least r[1] and r[2] such that

$$glt(r[1]) = glt(r[2]) = q$$

where $q < r[1] < r[2]$. But by the density of Q, there is a q' such that

$$q < r[1] < q' < r[2].$$

It could not be true in that case that $|R| > |N|$.

It is easy to see that such a proof is only excluded by the restriction of rhetoric to the rationals and the exclusion of the explicit image of the irrational. For our mathematic to be consistent we must keep the alogos indefinite.

Historically, mathematicians have argued themselves into accepting the irrational as number by claiming that it behaved as number when used in this or that established calculus. This is false. No irrational number has ever been subject to the operations of mathematics. It has rather been the case that symbolic representation of irrationals in some cases and the approximations of irrationals in other cases were consistent under the operations of number. This shows the consistency of the calculi of roots, series, or sequences and nothing of the irrational number.

Using N as a number, as Cantor does with 2 to the Nth power, leads to further contradictions. Consider the elements of C that are made up of repeating patters of decimals. These are all elements of Q as well. With one decimal place there are 10 repeating decimals. Two decimal places allow for ten as well:

$$0.00, 0.11, 0.22, \ldots, 0.99.$$

With four places there are one hundred repeaters; with six, one thousand. In general, for even n there are ten to the (n/2)th power of repeaters. As the number of decimal places goes to infinity, or N, there are 10 to the N/2 combinations of repeating decimal patterns.

$$10^{(N/2)} = (10^N)^{(1/2)} = (4^N * 2.5^N)^{(1/2)} = 2^N * (2.5^N)^{(1/2)} \geq 2^N.$$

And from this we have:

$$|Q| = |N| \geq 2^N = |R|.$$

The principle, the essence, of mathematics requires that if we accept Cantor's contribution to mathematics (these so-called power sets arising from the grammar of x to the y), we must accept their use wherever this grammar naturally arises in any mathematic. We must allow Cantor's rhetoric to pass consistently into the rhetoric of mathematics as a whole. And yet this would seem to be unacceptable.

The paradoxes of set theory and its various stepchildren, such as topology, lose their entertaining aspect in the light of this clearer picture of infinity. The rhetoric of mathematics in the twentieth century regarding infinity has been without principle. The contradictions which have arisen outside the bounds of reference have been upheld as paradoxes revealing the depth and majesty of mathematics. Those inconsistencies arising within the bounds of reference are merely posted with "Tyger" signs and ignored.

Even as mathematicians accept the current rhetoric, they often seem aware that all is not well. The tone of mathematical writing changes in the proximity of Tygers. In the presence of the alogos, the prose turns cautious, tentative, apologetic. Authors will drop the imperial we and, assuming the humble aspect of the first-person singular, escort us (blindfolded) around the hole in the mathematical mind. It is clear our authority is limited in Tyger Land.

If we are to clearly understand the mathematics of infinity and bring to it a principled expression, we must admit to ourselves that all the foundation work of Russell, Cantor, Weierstrass and the like has not sufficed to fill the hole we inherited from the Greeks. It is with a desire for this clarity that we turn now to Cantor's Diagonal Proof. By understanding what Cantor has given us, we will begin to see the form of the hole in our mind.

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3. Cantor's Diagonal Proof

Cantor's Diagonal Proof, the CDP, is his proof that the real numbers are not denumerable. This is to say the reals cannot be put into a one-to-one correspondence with natural numbers. His proof relies upon the form of the irrational number. By understanding the form and limitations of Cantor's instrument, we will be led to the underlying basis of the contradictions inherent in the form of number. Rather than filling the hole in the mathematical mind, this critique will bring the nature of this hole into the light of understanding.

Cantor's Theorem

The real numbers are not denumerable.

Proof

Let us list the natural numbers in their entirety. To each n in N let there correspond a distinct real number r in C, the interval (0, 1) that is equivalent to the set of real numbers as a whole. To avoid duplication, no r ending in an infinite repetition of the digit nine will be used. This is necessary because all numbers of that form (0.15000..., 0.14999...) represent the same number under the identity transformation of decimal conversion using the division algorithm (plain long division).

> Our list then is
> 1 0.a[1]a[2]a[3]a[4]a[5]...
> 2 0.b[1]b[2]b[3]b[4]b[5]...
> 3 0.c[1]c[2]c[3]c[4]c[5]...
> 4 0.d[1]d[2]d[3]d[4]d[5]...
> . .
> . .
> . .
> k 0.n[1]n[2]n[3]n[4]n[5]...
> . .
> . .
> . .

The diagonal of this list are the numbers di or 0.a[1]b[2]c[3]d[4]...n[k]...

We then consider the number g such that g[i] is not equal to d[i]. Clearly, g is not on our list because its first digit is not a[1], its second is not b[2], and its kth is not n[k]. Thus the list of real numbers paired with the naturals is incomplete. The reals cannot be put into a one-to-one correspondence with the naturals. Thus, the reals are not denumerable.

This proof is often described as showing how we may construct a number that is not an element of a denumerable set. This is not the case. Such a construction would be a sequence of rationals, the limit of which is not on any constructible list. And so any connection in our thought between the CDP and constructivism must first be thrown out before we may see clearly what is before us. The power of the CDP arises from applying the logical NOT to the diagonal of the list. The operation of NOT(d[i]) gives rise logically, all at once so to say, to a set of 9 to the N numbers. For each digit of g may be any of the nine digits it currently is NOT.

The relation of the finite to the infinite gives rise to another problem if we consider the CDP as some sort of a construction of a number not on the list. At each step of such a construction Cantor's number g is itself a description of the constructed number's place in a normal list of the natural numbers. At no point in the series would the constructed number depart from its membership in the rationals.

There is no constructive element in the CDP. All of the natural numbers are used in the list as indices, not merely arbitrarily many of them. And each natural corresponds to a real number that, if the proof is to have any meaning at all, is explicitly defined as having a particular digit in each of its infinite decimal places. The CDP can only address the reals and the naturals by addressing them in their entirety.

And yet the CDP is merely a picture suggestive of an entirety. The suggestion is strongest when the irrational numbers composing the list are indefinite ones. This is to say they must be conceived of as rows of random digits. We are to imagine a random list of infinitely long numbers each of which is in itself unknown to us.

Cantor's list is essentially an N x N matrix, a square of rows and columns equal in number to the cardinality of the naturals. If we begin to consider what forms such a matrix may take we soon see that the CDP relies heavily upon the indeterminate nature of the irrational as is appears in the current rhetoric of mathematics. If the matrix is created with patterned numbers then it is a list of rationals and the CDP is rendered meaningless. But we may consider certain patterns of entries for the matrix that begin as rationals and then become random.

Consider this matrix. Begin the first number with zero, the second with one, and continue th the pattern 0, .. 9, 00, .. 99, 000, .. 999, 0000 .... And let the digits following these patterns be arbitrary. The matrix then takes the form:

1 0.0a[1]a[2]a[3]a[4]a[5]...
2 0.1b[1]b[2]b[3]b[4]b[5]...
3 0.2c[1]c[2]c[3]c[4]c[5]...
. .
. .
. .
10 0.9d[1]d[2]d[3]d[4]d[5]...
11 0.00n[1]n[2]n[3]n[4]n[5]...
. .
. .
. .

Reorder the matrix so that the beginnings are

0 ...
10...
210...
3210...
....

The diagonal of the matrix is all zeroes. Below the diagonal of ones, another of twos, and so forth through all the natural numbers.

The d[i] of this matrix according to the CDP and its use of the zero diagonal is an irrational number made up entirely of the digits one through nine. If we permute the first row to the fleeing end or, for the squeamish, simply remove it, the new diagonal consists only of ones and the new di must consist of only twos through zeroes so long as they are not all zeroes. We repeat this operation N times and show that no number can be created that is not on the list. Remark that Cantor's proof has in no way been tampered with as it does not claim to rely upon the ordering of the elements of its list. Another way to look at this is to see that given such a matrix and a Cantorian d[i], the matrix can be permuted in a deterministic way to produce that d[i] in the diagonal and what exists in the diagonal must exist in a row.

But you may say, all such rows are patterned. Indeed they are, but the patterns are arbitrarily large. What Cantor gave us is a picture that assures us that we may step outside of these infinite patterns. If for each choice of a different digit, we know that the new number is in the matrix, at what point do we leave the matrix. Why do we feel differently about this picture when Cantor allows us to choose the new number all at once?

Rather than claim that this is a counterexample and refutation of Cantor's proof, this critique will claim that neither confirmation nor refutation of infinite form can come from a finite picture. Any proof that does not openly show the form of the object, or the doing of the refutation is offering us a mere picture. When the picture is outside the realm of demonstration and verification, our acceptance of the picture is guided solely by our preferences as to the unestablishable consequences of the generative axioms. That more or fewer thinkers share a preference does not speak to the validity of any picture. And as we have seen, questions having to do with preference or aesthetics are the generative propositions of mathematics and outside the realm of proof.

It is remarkable that the CDP was accepted, for two reasons. It is astonishing that a picture proof was accepted by a mathematical culture that eschews picture proofs. The culture that gave rise to the mathematic of the Indian subcontinent accepted picture proofs. But our Faustian culture came to reject picture proofs in favor of those springing from the form of Aristotelian logic, the logic embraced by the schoolmen that formed our culture. This logic of appeal to necessity in time assumed the form of the language of formal logic.

It is further remarkable that a proof that purported to be a construction was accepted by a mathematical community that was in the very process of rejecting constructivism. For had the mathematics of this century accepted construction as a basis, today's rhetoric would be built not upon the logic, formalism and set theory of Whitehead, Peano, and Hilbert, but upon the ideas of Brouwer.

We must distinguish between construction as an aspect and construction as a basis. Inductive proofs where an assertion is true for some small n0, assumed true for an arbitrary n and shown true for (n + 1), rely upon principle, not upon construction. Construction is an aspect of such proofs in as much as construction is an aspect of the progression of the natural numbers. This is to say that if we are given any picture based upon the natural numbers, we can construct the subsequent picture.

Cantor's proof is neither inductive nor constructive. We will see, in the next section, that it is presumptive. We must have our NOT(diagonal) all at once and in the form presumed by the proof itself. It is a fault in the CDP that it relies upon the effect of a picture. It is not a proof showing how some state is achieved or demonstrated false. It suggests what is to be believed. Mathematical proof takes place in a space disjoint from belief. Beliefs, in the form of generative propositions, form and define the "space" of a mathematic. Were our beliefs different, so would our mathematic have a different form. Mathematics is the sum of the tautologies arising from the consequences of our beliefs. We may consider our beliefs to be a transformation from the only world into human consciousness. Mathematics is the formal subspace of this transformation. And contradiction and paradox constitute the null space of our mapping.

The picture of the CDP is an arrangement of the problematic elements of the form of number in such a way that unless we wish to engage those problems where so many of our betters have failed, we must accept the picture we are offered.

Let us consider again the list above that used the naturals as a beginning for each element. The fist digit of the first number was zero followed by N digits. The second was a 1 and so forth. Any list used in Cantor's proof may be made into one by reordering it and adding the missing elements. And no such conformance would alter the list as N + 1 is always N. One may neither construct nor show the arisal of a number d[i] not upon this list. For each d [i] shows its place upon the list. Each digit inspected continues to correlate d[i] to a number and no such number is absent as the list is infinitely, not arbitrarily long. All of the naturals are included in the beginnings. Because no d[i] passes from a definite number to an indefinite number, d[i] must be on the list.

Just as one may take the power set of N, one may permute a list of numbers N times. Doing this and accepting Cantor's picture, we may permute our list once for each element on the list so that we have a Cantor d[i] contradicting that element's existence in the list. Hence, according to Cantor, one may show that no element upon the list is on the list in addition to all d[i] being on the list. This Tyger we have by the tail is possibly a generative proposition. If so, we must accept that neither accepting nor rejecting the picture it presents is provable, as the generative propositions lie outside the realm of proof. Unfortunately, we shall see that even this is not the case.

In all of this our difficulties arise only when we introduce the form of the natural numbers in their entirety. When we eschew every aspect of construction and accept the entire form of the natural numbers, we are led to a list for which the CDP fails. This suggests that Cantor's proof is not about lists but about our presumptions concerning the form of number.

The CDP works by eliminating the natural numbers in their entirety.

The contradictions in the form of number arise from the possibility of two conflicting pictures of the natural numbers and their infinite form. And mathematics has chosen to accept from place to place this or that of these two pictures rather than to choose one and exclude the other. They are:

1) Infinity is the inclusion of all members of N. If this is the case, then our list above will not work for the CDP for no number arises that is not on the list of natural beginnings.

2) Infinity is the open potential of infinitely many more. This is to say that given any finite set S of N there is an infinite set T = N - S. From this standpoint our list, no matter how it is constructed, lacks infinitely many natural numbers, all described by the NOT(diagonal).

But the concept of a set of all subsets requires the closure of N. One must have the totality so that expressions such as 2 to the N are meaningful. Then for case 1, the object 2^N arises which is greater than N and the list that does not uphold the CDP contains the property that upholds the claim of the CDP. While in case 2, 2^N is rendered meaningless. And so the list that upholds the claim of 2^N having a greater cardinality than N, contains the seed that undermines that proof.

A fundamental failing of the CDP is its basis in set theory. Or rather, the failure is of the underlying presumption that set theory is an explanation of or a foundation for number and mathematics as a whole. Regarding this, Wittgenstein once remarked that the lighthouse in a painting was in no way supported by the painted rock beneath it. One may view mathematics as a set-theoretic object. But only a mathematic that arises after such a standpoint is assumed can require set theory or be explained by it. Calculus does not require set theory. Neither does set theory explain the Calculus. But set theory has been a fertile standpoint for the creative expansion of the Calculus.

One must understand that set theory itself arose from certain questions of mathematics already in existence. It did not explain those mathematics but became itself a mathematic used for resolving some of those questions. Russell showed that set theory was unable to clarify the form of number. The Principia was essentially a restatement of arithmetic; and the Theory of Types, a renaming of the natural numbers.

The theory of transfinite numbers is the same awkward patch upon the form of number that Russell's theory had been. The paradox of the form of number is the relation of the finite to the infinite. And the difficult question has long been, "How does a continuance of the finite lead to infinity?" Cantor tried to establish the form of number upon sequences and their limits, but as an assumed basis and not the consequence of a basis. For such a system the CDP is unnecessary. That a number not be in a denumerable list becomes axiomatic (as it must indeed have been for Cantor).

But like the Theory of Types, the transfinites do not resolve the paradox. The transfinites place the paradox out of reach by renaming the naturals. The types and transfinites are the natural numbers. But now the problems of number take the form of the properties of a set of all sets of a certain kind or of the properties of uncountable sets beyond the reach of our concepts. Russell attempted to remove the alogos from mathematics and install it as a technical issue in the realm of logic. Cantor left the alogos in mathematics but placed it beyond our reach.

In spite of all their work, the hole is still here in our mind. Let us look at the substance of the alogos.

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4. Reason's Antinomy

The claim of this critique is that the natural numbers themselves are the hole in the mathematical mind. They are the underlying form of the contradictions that arise in all that we have so far discussed.

Another way to put this is to say that the set of the natural numbers is the underlying form of the antinomies of reason. Those unfamiliar with the antinomies are referred to Immanuel Kant's Critique of Pure Reason, especially the chapter dealing with these antinomies in Book II of the Transcendental Dialectic.

The purpose of this section of our critique is to establish the form of the natural numbers within the antinomies as delineated by Kant. We will then correct and amend Kant's estimate of mathematics and so establish the relation of pure reason, the mathematical mind, to the natural numbers.

Because familiarity with Kant's critique is necessary for an apprehension of this critique, that familiarity is taken for granted. Rather than a welter of footnotes, phrases set off by quotation marks are understood to be taken from the Norman Kemp Smith translation of the chapter, The Antinomy of Reason, in:

> Critique of Pure Reason
> Immanuel Kant
> St. Martin's Press, 1965.

Clearly the concept of the natural numbers falls within the realm of the antinomies, for they are the simplest example of an unconditioned unity of the "objective conditions of the possibility of objects in general." Kant's antinomies have to do not with "the absolute totality in the synthesis of appearances" but with the totalities as ideals of pure reason. Reason takes the concepts, from which the antinomies arise, from understanding and attempts to free them "from the unavoidable limitations of possible experience, and so to endeavour to extend it beyond the limits of the empirical, though still, indeed, in terms of its relation to the empirical." Reason demands that if the conditioned is given, then the entire sum of the conditioned. This, in turn, implies that the effect or consequence of the unconditioned totality is given. That the unconditioned has an effect or consequence is necessary so that the extension to the entire sum may remain in terms of the empirical.

Only those categories of reason are susceptible to antinomy "in which the synthesis constitutes a series of conditions subordinated to, not co-ordinated with, one another, and generative of a conditioned." Remark here that the form of a series, each element of which is subordinate to another and that generates a conditioned, can be no more clearly expressed than in the unity of successive elements composing the natural numbers.

Kant asserts, first, that "absolute totality is demanded only in so far as the ascending series of conditions relates to a given conditioned. It is not demanded in regard to the descending line of consequences nor in reference to the aggregate of co-ordinated conditions of the consequences." Second, he asserts "the question as to the totality of a series is not in any way a presumption of reason." Ascension in Kant is motion in the direction of the "1" in the series: 1, 2, 3, ... Descent is motion towards the "and so on." Ascent is thus given in its entirety; descent is thought of as "allowing of being given."

Antinomies apply to series. And they apply where complete comprehension of the elements, and thus their totality, requires knowledge of the grounds and not the consequences.

When Kant states that the totality of the series is not presupposed by reason, he must be referring to the series of objects necessary to each antinomy. Else he is mistaken. For the existence of the totality of a necessary series that can be put into one-to-one correspondence with the natural numbers is always conceived of as existing within its own entirety. This entirety is presupposed by reason. The irrational number is the consequence that reason extrapolates to be the empirical intuition of unlimited succession. It is the form of the natural numbers expressed within a single instance of number. And this totality as the consequence of the same mode of thought found within the antinomies.

Kant's four concepts that are subject to antinomy are:

1. Absolute completeness of the Composition of the given whole of all appearances.
2. Absolute completeness in the Division of a given whole in the appearances.
3. Absolute completeness in the Origination of an appearance.
4. Absolute completeness as regards Dependence of Existence of the alterable in appearance.

The antinomies are our absolute failures in the exposition or explanation of appearance. Reason is trying to ground itself, to clean up, and rest in the unconditioned that it believes to be contained in the absolute totality of a series. The idea of this completeness is accepted as apart from the possibility or impossibility of establishing any reference for it. Reason "leaves undecided whether and how this totality is attainable." And so we accept as an empirical intuition that which has no connection to the empirical.

Kant states that the "unconditioned may be conceived in either of two ways. It may be viewed as consisting of the entire series in which all the members without exception are conditioned and only the totality of them is absolutely unconditioned. This regress is to be entitled infinite. Or, alternatively, the absolutely unconditioned is only a part of the series--a part to which the other members are subordinated, and that does not itself stand under any other condition." The first, if we take the series to be the absolutely abstract form of all series, the natural numbers, is infinity as a whole, the closure of N. The second is infinity as infinitely many more, that series that has no closure and that imposes the zero distance upon us, with the form of the alogos outside the zero distance.

For Kant, the ambiguity is in the form of the unconditioned. And for him the antinomies arise as the concepts of pure reason are forced into an empirical reference that requires them to fill out the form of the unconditioned. Extending the principles of reason beyond the limits of experience "there arise pseudo-rational doctrines which can neither hope for confirmation in experience nor fear refutation by it." All of these doctrines are antithetical pairs of doctrines arising from the concept in question being put arbitrarily into either form of the unconditioned.

These two-sided dialectical illusions of reason, for like illusions they only give the appearance of reason and "relate not to the unity of reason in mere ideas."

Let us now consider the antinomies themselves. We will see that each pair of arguments rests upon the form of number, each argument a manipulation of the inconsistencies inherent in the alogos. In order that we may see this clearly, we lay before us the thesis of the first antinomy in its entirety that we may observe the surgical operation that lays open to view the form of number. Upon the antithesis of the first antinomy we will operate in a less tedious manner. And for the remaining antinomies we shall simply summarize their reliance upon the form of number.

Here then is Kant's thesis from the first antinomy:

Thesis

The world has a beginning in time, and is also limited as regards space.

Proof

If we assume that the world has no beginning in time, then up to every moment an eternity has elapsed, and there has passed away in the world an infinite series of successive states of things. Now the infinity of a series consists in the fact that it can never be completed through successive synthesis. It thus follows that it is impossible for an infinite world-series to have passed away, and that a beginning of the world is therefore a necessary condition of the world's existence. This was the first point called for in the proof. As regards the second point, let us again assume the opposite, namely, that the world is an infinite given whole of coexisting things. Now the magnitude of a quantum which is not given in intuition ^(a) as within certain limits, can be thought only through the synthesis of its parts, and the totality of such a quantum only through a synthesis that is brought to completion through repeated addition of unit to unit ^(b). In order, therefore, to think, as a whole, the world which fills all spaces, the successive synthesis of the parts of an infinite world must be viewed as having elapsed in the enumeration of all coexisting things. This, however, is impossible. An infinite aggregate of actual things cannot therefore be viewed as a given whole, not consequently as simultaneously given. The world is, therefore, as regards extension in space, not infinite, but is enclosed within limits. This was the second point in dispute. ^(a) An indeterminate quantum can be intuited as a whole when it is such that though enclosed within limits we do not require to construct its totality through measurement, that is through the successive synthesis of its parts. For the limits, in cutting off anything further, themselves determine its completeness. ^(b) The concept of totality is in this case simply the representation of the completed synthesis of its parts; for, since we cannot obtain the concept from the intuition of the whole--that being in this case impossible--we can apprehend it only through the synthesis of the parts viewed as carried, at least in idea, to the completion of the infinite.

end of Kant

The first antinomy considers the question of whether the setting of our only world, that is time and space, is or is not infinite. The form of the question imposes upon the argument the necessity of introducing a countably infinite set that can be placed in one-to-one correspondence with the natural numbers. And indeed, in the first sentence there appears "an infinite series of successive states of things." Now an argument within an antinomy must assume one of the two contradictory standpoints regarding the form of number. It must assume either that the natural numbers compose an object that may be taken in its entirety or that the natural numbers are the expression of having no delimitable entirety, of infinitely many more. And in the second sentence the second standpoint is assumed, "the infinity of a series consists in the fact that it can never be completed through successive synthesis."

In the second part of the thesis again an infinite set is introduced, "assume second part turns upon the established finity of time, its underlying standpoint is again that of infinity outside of the possibility of completion, that which imposes the zero distance upon us." The argument then continues that the cardinality of this set, the magnitude of a quantum which is not given in intuition as within certain limits, can only be established as infinite "through repeated addition of unit to unit." This assertion shows the form of the argument identical to the form of the natural numbers.

It is not the place of this critique to pursue the arguments of the antinomies. It is rather to establish the presence of the form of number within the antinomies. One sees in the thesis of the first antinomy that the form of number in its aspect of the alogos is not merely present but dominant. But it is more than dominant; it is necessary. We will see that each thesis and antithesis is but the contradictory form of number in dress appropriate to each transcendental idea.

Remark that of the notes used to clarify the proof of the thesis, the first is simply the expression of the form of the irrational number. The second emphasizes that infinity is not a synthesis but the act of conceiving a recursive synthesis as completed.

Let us consider the first antinomy's antithesis:

Antithesis

The world has no beginning, and no limits in space: it is infinite as regards both time and space.

Proof

For let us assume that it has a beginning. Since the beginning is an existence which is preceded by a time in which the thing is not, there must have been a preceding time in which the world was not, i.e. an empty time. Now no coming to be of a thing is possible in an empty time, because no part of such a time possesses, as compared with any other, a distinguishing condition of existence rather than of nonexistence; and this applies whether the thing is supposed to arise of itself or through some other causes. In the world many series of things can, indeed, begin; but the world itself cannot have a beginning, and is therefore infinite in respect of past time. As regards the second point, let us start by assuming the opposite, namely, that the world is finite and limited, and consequently exists in an empty space which is unlimited. Things will therefore not only be related in space but also to space. Now since the world is an absolute whole beyond which there is no object of intuition, and therefore no correlate with which the world stands in relation, the relation of the world to empty space would be a relation of it to no object. But such a relation, and consequently the limitation of the world by empty space, is nothing. The world cannot, therefore, be limited in space; that is, it is infinite in respect of extension.

end of Kant

This antithesis begins with an empirical, causal argument identifying time with the form of infinity as incompletable. By arguing that no non-element of a series can cause the first element of the series to arise, it identifies our relation to time as identical with our relation to a series within a zero distance that does not include the origin. Upon the real line, this would be any countable series within an open interval that does not include zero. By denying the first element of a series one necessarily identifies the series with infinity and the zero distance.

The second argument of the antithesis takes the same standpoint regarding infinity. For Kant, space is identical to the form of outer intuition. Our relation to outer intuition is that of a countable series that can be said to be only accidentally bounded by death in the case of the individual. And so the series may be considered as having no necessary upper bound when abstracted from the individual. Space then, a priori, takes the form of incompletable infinity. And it is only left for the argument to show the world infinite by showing intuition to be non-empty.

We see then that the alogos is necessary not merely to the arguments of the antinomies but to the underlying presumptions of the arguments. Without the form of number, the antinomies evaporate. This is not to say they would be resolved that way. Rather, without the contradictory form of the unconditioned as object, they could not be formulated.

The form of number in the second antinomy is best apprehended if we consider the form of the theory of transfinites. Recall that the transfinites begin with the countable set of the natural numbers. The next transfinite is the set of all the subsets of the naturals. The third is the set of all the subsets of the second and so on. This recursive form must rest upon a primitive countable set.

Now consider the second antinomy. This turns about whether the composite is reducible to the simple. The thesis takes the standpoint that infinity has closure and presumes the composite to be made of a series of composing elements. By considering only composite substances in themselves we are left with the series of elements themselves as the ultimate ground. Building up a concept of composites is a recursion, as Russell shows in his construction of the Theory of Types. No matter how far the recursion is carried, even to infinity, one may not ascend beyond the primitive countable set. And in the thesis this primitive countable set must reside within the composition.

The antithesis takes the same standpoint regarding infinity and the same standpoint regarding composition. It merely concretises the argument from abstract isolate composition to composition in space. We saw in the first antinomy that Kant regarded space as made up of countable subspaces. He places the composite in space in this way, "Space, however is not made up of simple parts, but of spaces. Every part of the composite must therefor occupy a space." The countable set of subspaces is thus made the primitive ground for the recursive decomposition. And so, in a sense, this argument uses the form of infinity as infinitely many more by placing the series of decomposition outside the zero distance of the primitive countable set.

Remark that it is in no way necessary that Kant have been in possession of anything like the theory of the transfinite numbers in order for the properties of such sets to underlie his arguments. For the properties of the transfinites are identically the properties of the natural numbers. It was necessary merely that Kant be dealing with the concept of a countable series and its totality. The contradictory form of the alogos is no more and no less than the immediate consequences of the concept of the natural numbers.

In the thesis of the third antinomy, Kant again establishes a countable series and then identifies it with an aspect of the alogos. The series is the chain of causality, "everything which takes place presupposes a preceding state upon which it inevitably follows according to a rule." He then identifies this series with infinity without closure. "The causality of the cause through which something takes place is itself, therefore, something that has taken place, which again presupposes, in accordance with the law of nature, a preceding state and its causality, and this in similar manner a still earlier state, and so on. If, therefore, everything takes place solely in accordance with laws of nature, there will always be only a relative and never a first beginning, and consequently no completeness of the series on the side of the causes that arise the one from the other." The contradictory from of this aspect of infinity is used to prove the necessity of its opposite. In the antithesis, the forms are merely reversed, the contradictory nature of closure implying the necessity of non-closure.

At the end of the antithesis, Kant writes, "The illusion of freedom [which requires the closure of infinity], on the other hand, offers a point of rest to the enquiring understanding in the chain of causes, conducting itself to an unconditioned causality which begins to act of itself. This causality is, however, blind, and abrogates the rules through which alone a completely coherent experience is possible" We may add, in our abstract interpretation of the antinomies, that the closure of infinity, which wraps the head of the natural numbers back around to its tail, is also a resting place for the inquiring mathematical mind and one that abrogates the rules through which alone a completely coherent form of number is possible.

The thesis of the fourth antinomy establishes a series and chooses an aspect of the alogos, the contradictory nature of which is used to prove the argument. In the antithesis, both aspects of the alogos are used, "Either there is a beginning in the series of alterations which is absolutely necessary and therefore without a cause, or the series itself is without any beginning, and although contingent and conditioned in all its parts, none the less, as a whole, is absolutely necessary and unconditioned." The argument is then proven by using the contradictory nature of both aspects, "The former alternative, however, conflicts with the dynamical law of the determination of all appearances in time; and the latter alternative contradicts itself, since the existence of a series cannot be necessary if no single member of it is necessary."

In 1918, Ludwig Wittgenstein wrote of his Tractatus,

> The book will, therefore, draw a limit to thinking, or rather--not to thinking but to the expression of thoughts; for, in order to draw a limit to thinking we should have to be able to think both sides of this limit (we should therefore have to be able to think what cannot be thought).
> Tractatus Logico-philosophicus, p.27
> Ludwig Wittgenstein, RKP, 1922.

Wittgenstein was the first to truly harrow the ground broken by Kant. The Tractatus was the first serious reckoning of the consequences of the necessities of the Critique of Pure Reason.

The Tractatus is mentioned here in part to point the mathematician to the beginning of a body of work that considers in depth the foundation and reference of mathematics. It is mentioned as well in order to show that Kant too was concerned with the limits of thought.

Kant writes, "The question, therefore, is whether in transcendental philosophy there is any question relating to an object presented to pure reason which is unanswerable by this reason, and whether we may rightly excuse ourselves from giving a decisive answer." Having shown us the antinomies, Kant begins to delineate what is left to us after this illusory dialectical ground has been cut from beneath our feet. "Although to the question, what is the constitution of a transcendental object, no answer can be given stating what it is, we can yet reply that the question itself is nothing, because there is not object corresponding to it."

The transcendental objects are one and all the alleged consequences of a synthesis of objects which may be put into one-to-one correspondence with the natural numbers and which therefore are subject to the inherent contradictions that arise from the form of number. "Our sole question is as to what lies in the idea, to which the empirical synthesis can do no more than merely approximate; the question must therefore be capable of being solved entirely from the idea. Since the idea is a mere creature of reason, reason cannot disclaim its responsibility and saddle it upon the unknown subject." This object in each antinomy is the absolute unconditioned totality of a synthesis of a series of appearances. We are holding up as an object, empirically referenced, that of which "we should require what is not possible through any empirical knowledge, namely, a completed synthesis and the consciousness of its absolute totality." Such an object cannot ever be an object of understanding but must remain a postulated existence outside the realm of reference.

And postulating such an existence explains nothing, "We should not, for instance, in any wise be able to explain the appearances of a body better, or even differently, in assuming that it consisted either of simple or of inexhaustibly composite parts: for neither a simple appearance nor an infinite composition can ever come before us." By choosing between two groundless pseudo-syntheses nothing in understanding is altered or improved.

"The critical solution, which allows of complete certainty, does not consider the question objectively, but in relation to the foundation of the knowledge upon which the question is based." The transcendental objects, each an individual expression of the underlying form of the totality of an infinite series, arise from the attempt to answer a question grounded in the desire to tidy up and explain the context of reason itself. We should first ask "whether the question does not itself rest on a groundless supposition, in that it plays with an idea the falsity of which can be more easily detached through study of its application and consequences than its own separate representation."

Kant shows that in all concepts that lead to antinomies manifest a similar form, "If therefore, in dealing with a cosmological idea, I were able to appreciate beforehand that whatever view may be taken of the unconditioned in the successive synthesis of appearances, it must be either too large or too small for any concept of the understanding, I should be in a position to understand that since the cosmological idea has no bearing save upon an object of experience which has to be in conformity with a possible concept of the understanding, it must be entirely empty and without meaning: for its object, view it as we may, cannot be made to agree with it. This is in fact the case with all the cosmical concepts: and this is why reason, so long as it holds to them, is involved in an unavoidable antinomy."

The common element in the cosmical concepts is the form of number. And so it must be said that the unconditioned totality of the natural numbers, the set N, in all its manifestations, is at once too large and too small for any concept of the understanding. From the standpoint of counting, of constructivism, the natural numbers are too small to fill the necessary form of number. From the standpoint of infinity with closure, the standpoint of the transfinites, of the alogos, the natural numbers are too large for the understanding, passing beyond the bounds of reference.

"The whole antinomy of reason rests upon the dialectical argument: If the conditioned is given, the entire series of all its conditions is likewise given. ... In the first place, it is evident beyond all possibility of doubt, that if the conditioned is given, a regress in the series of all its conditions is set us as a task. ... Further, if the conditioned as well as its condition are things in themselves, then upon the former being given, the regress to the latter is not only set as a task, but therewith already given." Number is not a thing in itself. It is an imposition of the mind upon the world, an idea. And the regress that number sets us as a task adds nothing to the idea. The synthesis of all the conditioned is a question of our knowledge so each identity of our construction of the suggestion of some whole. And it is construction that underlies the set of natural numbers. For the antinomies result from beginning with a transcendental category and the claim of having accomplished an absolute empirical construction. In the abstracted form of the natural numbers, this is not an empirical construction. It is a construction that is referenced throughout the synthesis to the empirical in one-to-one correspondence. And so in its completed unconditioned form, the natural numbers are a necessary claim of reference into the empirical. Number is the consequence of serial intuition. Yet none of this requires us to decide upon the form of the naturals as a whole.

In this region of thought governed by the form of number "two dialectically opposed judgments both may be false; for the one is not a mere contradictory of the other, but says something more than is required for a simple contradiction." We have seen that the dialectical opposition of judgements is only possible when we are judging the form of number. We might say the validity of infinity is not to be found as an axiomatic totality but as the formal problem of the understanding in relation to the natural numbers.

When one says, "Consider the natural numbers....," the grammar is the same as that of "Consider a region without boundaries that begins here...." There is no object in thought that can be referenced to either of these. All reference is bounded. and the unbounded, whether a region or a set, can be no more than a setting for thought, a stage for referenced objects. Further objects are judged as either appropriate or inappropriate to a given setting. Individual and determinate numbers are empirical objects. You can think no more about a number than you can physically represent or write down. The set of naturals, the unbounded region, is a transcendental object.

Kant believed mathematics to be exempt from the antinomy of reason. He writes, "But in mathematical problems there is no question of this [possible experience that he claimed to be at the heart of the antinomies], nor indeed of existence at all, but only of the properties of the objects themselves, solely in so far as these properties are connected with the concept of the objects." But the world of possible experience and the world of reference are the identical and only world that we are given. And the natural numbers as a whole are the primitive and primal transcendental idea which has "a purely intelligible object, but only if we likewise admit that, for the rest, we have no knowledge in regard to it, and that it cannot be thought as a determinate thing in terms of distinctive inner predicates. As it is independent of all empirical concepts, we are cut off from any reasons that could establish the possibility of such an object, and have not the least justification for assuming it. It is a mere thought entity."

Kant distinguishes between two types of antinomies. The first two antinomies he terms mathematical and may be seen as reliant upon the constructive nature of the series. The second two antinomies he terms dynamical and are concerned with existence as a consequence of a series. Error arises in mathematics with respect to the form of number when reason reaches beyond the zero distance. Beyond that point no intuition of the properties of objects are possible and their predicates are indiscernible. Neither by construction nor by claim of existence may anything meaningful be said of that which lies beyond the limit of the expression of thought.

The method Kant used to discover and explicate the antinomies of reason "may be entitled the skeptical method. It is altogether different from scepticism--a principle of technical and scientific ignorance, which undermines all knowledge, and strives in all possible ways to destroy its reliability and steadfastness." Referring to mathematics and the skeptical method, he continued, "In mathematics its employment would, indeed, be absurd; for in mathematics no false assertions can be concealed and rendered invisible, inasmuch as the proofs must always proceed under the guidance of pure intuition and by means of a synthesis that is always evident." Or as Wittgenstein (and St. Paul) said: Nothing is hidden.

It is not the place of this critique to argue the value of Kant's work in the larger philosophical context. Nor is it our place to restore those ideas of value that were thrown out when academicians threw out his Euclidean dogmatism. But we must address his devaluation in mathematics. It was the prestige of Kant's work that stood against the acceptance of the important and necessary idea of non-Euclidean geometry. Kant's work has assumed a status similar to that of Aristotle's in medieval times. In certain spheres of thought, arguments were reduced to appealing to Kant's authority instead of to reason, the latter being the harder work.

Mathematics in Kant's time was in many ways not far removed from the mathematics of Alexandrian times. The axioms of mathematics, especially of geometry, were accepted as self-evident empirical assertions. It was Kant's viewpoint that space did not exist of itself. It was neither a physical nor a metaphysical reality. Space was only the form of outer intuitions. Having established this he mistook the apodeictic nature of Euclidean geometry for a description of this form. This lead him over the years to reify, or rather re-reify, space and eventually declared absolute space a necessity. Which is not at all his standpoint in the Critique of Pure Reason.

His original concept of space did not rest upon Euclidean geometry. Space in the Kantian sense is a primitive and so outside understanding. We cannot comprehend the form of our outer intuition. We are on the receiving end of it; it is simply given as a generative axiom of Kantian understanding.

Yet when we consider space as a concept apart from its role in the world, we stand in relation to it exactly as we do with any other concept. As subject to number, space is subject to our formal manipulations. Non-Euclidean geometries merely showed there to be more than one way to reduce space to continuous quanta. Had Kant known of Lobachevskian and Riemannian spaces, he might have accepted them as variant interpretations of space, each with its own a priori necessities, without altering his critique.

And so for Kant, because the basis of the axioms in the world was unquestioned, mathematics was the successful employment of reason beyond the limits of experience. But analysis through mathematics is not a reaching beyond experience. It is a sifting of experience's salient features. The pondering of experience is not outside experience.

Mathematics fails when detached from experience, when it passes beyond the bounds of reference. This is to say that, regarding infinity, mathematics cannot choose either standpoint offered by the alogos. To choose either is to prejudice all the models based upon mathematics unnecessarily. Any consequence of a model that relies upon such a choice is not wrong, it is nonsense. For such a model is equivalent to a model relying upon the contradictory form of the alogos, the form not chosen. The form of number being the primitive antinomy of reason, the form of its unconditioned totality is undecidable. Where no choice has meaning, none can be made.

We are all prisoners of the form of reason and, also, of the form of human will. It is easy to see why those who made mathematics have chosen the pictures of infinity, of the irrational, of the form of number that they did. It is surprising that those like Hilbert and Lesbegue resisted this choice as well as they did. For in accepting the current form of number, we have entered "a sphere in which it is no longer necessary for [the understanding] to observe and investigate in accordance with the laws of nature, but only to think and invent, in the assurance that it cannot be refuted by the facts of nature, not being bound by the evidence which they yield, but presuming to pass them by or even subordinate them to a higher authority, namely, that of reason ... indolence and vanity combine in sturdy support of these principles."

It is hard to resist temptation.

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5. In the Wake of Occam's Razor

There is the standpoint of understanding a given calculus. This is accepting the calculus as it is given us and following it obediently. Then there is the standpoint of seeing how a calculus has arisen and the context from which it arose. This is to stand outside the calculus. Only from outside the calculus can we see what a calculus fails to address in its context. The present calculus of infinity fails to address the contradiction inherent in the form of number. It chooses to ignore the alogos.

The present calculus of infinity is essentially that of Cantor. Most of what is held true of the form of infinity, including the form of the real numbers, depends upon the ideas of Cantor. We may think of Cantor's propositions as generative. Such generative propositions affect only their descendants. It is beyond cavil that the natural numbers are the primitive form of the antinomy of reason. And what is denied the transcendental concepts must be denied to their common form and descendants.

What is denied to reason is the form of infinite number. Whatever is outside the zero distance cannot be thought. And no axiom that attempts to supply a form for infinity can avoid either the contradictions inherent in number or the false consequences that must arise in any structures built upon the premise of such a form. For if we accept such a for as a true state of affairs, the antithesis of such a form is equally a true state of affairs. If the development of a thought breaks down in the working out of its consequences, the destructive nature is not to be found in the nature of the work--for the workers are worthy--but in the pre-existent basis of the thought.

Cantor presumed to delineate a form for infinity. This form has been so embraced by mathematicians that the descendant's of Cantor's propositions permeate analysis, set theory, topology, and many other fields of mathematics. The current form of infinity, the accepted form of number, has a basis that is equipollent to its contrary. It has no basis at all.

For half a century, mathematics has made the claim that mathematics needs no other basis than tautological construction upon arbitrary axioms. This may be thought of as a plea to be excused from the only world. For the form of number, like the forms of space and time, is inherent in the form of consciousness as we are given it. The common form of consciousness is the form of our only world. And no abstract and formal calculus within that consciousness has a meaning apart from reference that bears upon this world. Meaning is the relation of reason to reference.

We must understand clearly and finally what is the primitive antinomy of reason. Its descendants, delineated by Kant, were the nilpotent and mutually annihilating arguments that the world does and does not have a beginning, that is does and does not have a smallest part, that freedom and God do and do not exist.

The primitive antinomy of reason is that the set of natural numbers does and does not include itself.

This is to say that we may take the natural numbers as a completed object and, through recursive application of mathematical grammars, use it to create larger forms of itself. Such are the theories of types and of the transfinites. These theories substitute a complex recursion for the simpler recursion of: If n then n plus one. In all these the first element is the last non-recursive element, the second the first recursive one. The appeal of the transfinites is that it makes the entirety of N the last non-recursive, that is to say the last element that is no trouble to reason, and places the troublesome element of the set of the natural numbers beyond the reach of grammar.

Kant has shown us that all arguments based upon antinomies are meaningless. It remains to this critique to sketch the extent of meaning, to delineate the entities that should not have been multiplied and to survey what remains.

In the final section, we will identify the current basis of mathematical thought that accepted the current form of number and suggest a more mature basis.

Those mathematical objects are meaningless that rely upon one aspect or the other of the primitive antinomy. The picture of the real numbers must be cleared of the shadow of the alogos. It is beyond question true that the real numbers that are irrational require a representation in decimal form with decimal places in one-to-one correspondence to the natural numbers. What the antinomy precludes is the inference of anything further from this state of affairs. We may not treat these irrational numbers as if their completed form were within our understanding's grasp.

Regarding the real numbers, we must admit that we have all fallen foul of the antinomy. Mostly we have simply done what schoolmen of all Ages have done: received the Truth from Great Names and proceeded to attempt the addition of arabesques to the work of better men. Having followed Cantor, we have taken the consequences of Cantor for granted. But even in recent times some mathematicians have fallen afoul of the other side of the antinomy. Kronecker flatly denied the existence of irrational numbers. When we consider the value of his creative mathematics we have difficulty in reconciling his work with his view of number. That is, unless we assume that Kronecker found the undeniable nature of the arguments opposed to the accepted picture of those irrationals. When one believes that one has found a counter-example to what is accepted, the natural consequence is to feel that the converse of what is accepted must be true. Yet upon the ground of the antinomy, we are all tautologically wrong.

Brouwer, in his brief arc from twoness to obscurity, built his edifice of mathematics upon the antithesis of our current view. His motive was an extremely conservative one. Brouwer attempted to return mathematics to the basis of absolute empirical construction. His was a bizarre leap backwards into Hellenistic thought. Reading Brouwer always calls to mind Pythagoras. Where Kronecker threw out the irrational, Brouwer tried to construct them out of overlapping nests of rationals.

While it is true that we are prisoners of the zero distance, the consequences of the form of number are not. Any clear and necessary picture of the reals must extend beyond arbitrary measure into infinite measure. All numbers that have been reconciled to the operations upon number remain in the set of the reals. The antinomy merely requires that we give up any false claims as to our knowing anything about the form of number beyond the zero distance.

We must also give up the re-application of such false knowledge to set theory. Once one understands how the principled application of the form of the natural numbers undermines Cantor's Diagonal Proof one sees that there is no basis for any claim of a larger cardinality than that of the natural numbers. Nor can any such basis be formulated without recourse to the false ground of the antinomy. It is remarkable that mathematicians have so embraced the structure of the transfinites given the accepted failure of the identical form with different labels as found in Russell's Theory of Types. Perusing mathematical texts from Russell's time to our own, we see that for a time mathematicians acknowledged the failure of Russell's system and were concerned about the contradictions within mathematics.

Over time the texts turn slowly from Russell to Cantor and the sense of difficulty connected to the form of number passes away. This change is understandable. Mathematics currently claims to have merely arbitrary bases where Russell and Whitehead were firmly grounded in reference. Only in Cantor do we receive the comfort of an apparent escape from reference.

The antinomy prevents us from treating the natural numbers either as an open or as a closed set. We are left with no cardinalities beyond N. And we quickly find that nothing is lost by consigning the transfinites to the dust bin. No meaningful mathematics have been built upon those sets. In topology they have merely required a transfinite induction on the head of every normal one and have contributed to the creation of paradox. Among the mathematics using the transfinites may be found much that reveals an unfolding understanding of the form of number. These will be proofs that maintain their reference both to the natural numbers and to the referenced objects of topology.

It should not be thought that either the mere presence of the antinomy nor of abstraction beyond the bounds of reference negates any real mathematical ideas developed in a given work. This critique is by no means a plea to replace one dogma with another. Understanding the consequences of the antinomy and the necessity of reference clarify our standpoint but do not simplify our work. Instead they hold our work in mathematics up to a higher standard.

We will find that abandoning the meaningless causes no loss to the meaningful expression of mathematics. The continuum is unaltered by accepting reason's finite relation to the infinite. The apprehension of a continuum implies an absolute, infinite, transcendent, seamless concept. Continuity as well as completeness arises from the seamless principled expression of the natural numbers. The real numbers are those that lawfully arise from meaningful expressions having the reals as their range. There is no need for a completeness axiom, for there can be no gap in the expression of such a grammar. Viewed from outside, mathematics may appear incomplete or contradictory. From within, the space of mathematics is complete, finite yet unbounded. It is the space of our infinitely expanding understanding of formal principle.

We must see that the axiom of completeness accomplishes nothing. We must ask ourselves what it would mean to have a gap in the reals. Given the way in which the irrationals arise from the principled structure of the natural numbers, how would any ellision in the reals occur? Our reason can find no object that would represent such a gap. If, as Wittgenstein remarked, we have all of the reals except the square root of two, how would we know it was missing? Where, so to speak, would we find the place to insert it in the real line?

Because we can give no meaning to incompleteness, the completeness axiom serves no purpose except to fulfill an essentially Hellenistic desire that our construction be missing no part. It clearly makes sense to say, "There can be no gap in the reals." Yet there is no object in understanding to represent such a gap; and so the claim of completeness is without meaning.

There is another flaw in our idea of a gap in the real numbers. It arises as well in our rhetoric of functions. It is the common experience of undergraduates to hear that functions are defined by a set of pairs of real numbers. The first element of each pair is from the domain and the second is the first transformed into the range. Clearly, if the function is defined on an open set of the reals there are infinitely many pairs. No function may be defined by an infinite set unless we turn to metaphor as a basis of mathematics. It can only be true that a function defines the pairs and this only by assuming the aspect of a law or principle. The mistake here is that we place principle at the mercy of the confusion of language.

In the idea of completeness we again have an infinite structure of the reals. But in the rhetoric of completeness we allow the standpoint of the zero distance to intervene, to cast doubt on apprehension of the principles form of number, and cause us to fall back upon an axiom. Just as the rationals arise from the naturals, the irrationals arise in a principled way from the rationals. In some cases, the irrational number has been first apprehended through a relationship, geometric or functional, and only later shown to be expressible from the basis of number. But just as the principle of a function implies its set of infinite expressions, the sequential principle of the natural numbers, ascending without elision, implies, through the rationals, the completeness of the irrationals in their completion of the reals.

The idea that at the end of these series of infinities is some missing number is a consequence of Hellenistic constructivist thought, dragged into a context in which it has no meaning. The calculus of infinities is wholly apart from construction. In what other case do we fear that for some function, given an element of the domain, no element of the range will exist? What would be the meaning of such a fear?

When we put our cart before our horse and define functions by mystic sets or protect ourselves from gaps that we cannot even structurally conceive to ourselves, we are falling into pitfalls of language. Seeing these pitfalls for the mistaken beliefs that they are, clarifies our understanding of the objects of mathematics.

Completeness and continuity arise from the unity expressed in the relation of each adjacent pair of natural numbers. The principle underlying the latter implies the former.

6. The Basis of Principle

If the language of fundamental mathematics has remained vague and fraught with pitfalls, it is because mathematicians have allowed fundamental ideas to remain vague. Once the principle of the form of number is apprehended, the form's expression is clarified and language can no longer lead us astray. A critique is never an addition to the body of knowledge it critiques. It is always a reduction of that body through the elimination of illusory knowledge and of ambiguities of thought. When we see clearly, we see less.

In its domain, a critique destroys the false claim that the mind and its constructions have any hegemony over the only world. In as much as a philosophical critique is successful, it removes belief and anchors understanding. This understanding is the apprehension of the only world. And this world is not a mechanistic space of matter and motion. It is the space of understanding itself, which each of us interpret through the only objective lenses given us: the forms of space, time, and number. Such a description of the only world, like the mechanism of matter and motion, is again a reduction of the ethical and meaningful substance of the world. But in this critique we are speaking to mathematics, which is capable of only a formal reductionist representation of the world.

The current basis of mathematics is an expression of the underlying form of science. As science has become the moral basis and motive of thought in many cultures, it has brought its inherent weaknesses into mathematics. The two fundaments of scientific thought are progress in its linear accretive form and explanation in the form of models.

Wittgenstein wrote,

> Our civilization is characterized by the word 'progress.' Progress is its form rather than having the making of progress as one of its features. Typically, it constructs. It is occupied with building an evermore complicated structure.
> Culture and Value, p.7
> Ludwig Wittgenstein ,Chicago, 1980.

Progress is the justification of the present by fitting it into a self-justifying, artificial pattern of past activity. Progress is the justification of our gains and activity. In its present form, it is also our mental relation to history. Explanation is the maintenance of a moral standpoint, that of subjecting the world to our intellectual activity. Explanation is the justification of our acts, our mental relation to morality. Explanation is the perversion of representation, just as progress is the perversion of clarity.

Mathematics has become mesmerized by the combinatorics of axioms. And the axiomatization of mathematics, like the standpoint of relativity, is not the accomplishment of a new standpoint, but is the continuation of an old one. These standpoints are deeply conservative attempts to uphold outworn modes of thought. There are mathematicians trying to map algebra onto analysis in the attempt to show the equivalence of the two. This is like establishing some equivalence between a stone axe and a stone drill. Both are artifacts from a particular historical context made so that men could achieve some end. But we already know this of our artifacts without the necessity of proof.

The architecture built upon axiom is bounded and rises no higher than the implications of its basis. Its rooms, so to speak, are potentially very large. But the architecture is concrete, not potential, and the substance it expresses is limited and finite. This is more than a paraphrase of Goedel. A construction of TinkerToys will never be anything more than a collection of TinkerToys. Truth, the mapping of the only world onto our understanding, is inaccessible to combinatorics.

Explanation, by means of complicated constructions, seems possessed of a self-assurance. This pseudo-certainty is merely the effect of building an edifice of tautologies. Representation, on the other hand is more humble, more creative. It rests upon understanding which forms a deeper and more fluid basis than axiom.

Mathematicians practicing pure mathematics have renounced responsibility for their work having reference to the world. In their view, mathematics is responsible only to the formal basis and rules of each mathematic. But a formal language is a mechanism for exploiting one's present understanding of an object that exists outside that language in the only world. The language tells us all that we can know given the mechanism of the language and the current state of our apprehension of real states of affairs as caricatured by the language. This exploitation has traditionally been the domain of minor mathematicians. The primary creators of mathematics have concerned themselves not with exploitation but with the fundamental alteration and expansion of our apprehension of the world. Pure mathematic in our times has chosen to relegate itself to a minor and dependent role. Further, it claims that the master upon which it depends is dead, for with Poincare they echo that the age of mathematicians, like the age of miracles, is past.

We must recognize the necessity of both sense and reference to mathematics. Mathematics without reference can be likened to a formal study of Lewis Carroll's Jabberwocky. In such a study one is required to write about the poem ad indefinitum. Any reference to nonsense words, such as borogroves, would have to be in terms of other nonsense words. A scholar would have to use the nonsense words of his fellow scholars unless there seemed to be sufficient justification for introducing one of his own. All nonsense words would have to be non-referential, without connection to the world. So long as the work was not self-contradictory, it would be correct, even successful--leading to tenure.

Any such work could claim reference to the world in as much as it originates from the poem that is itself in the world. And the argument for reference might be strengthened by pointing out that the poem itself is full of reference to the world, if one excludes the nonsense words. But there is more to reference than a connectedness to objects. An expression in mathematics is valuable to the extent that it is connected to the referential expression of other mathematicians.

Because mathematics in its real context is not immune to the influence of science and politics (or, rather, let us say, of the defining ideas of its Culture), we must separate reference in this critique from any connection to these non-essential influences. It is impossible for a set of mathematical expressions to have a value due merely to the branch of mathematics to which the expressions belong or to the school where the expressor resides. The valid ideas of the Brouwer's in mathematics still have and always will have their essential place in mathematics even if our understanding is never great enough to reconcile them with the mathematics accepted in our time.

Also, the power and connectedness of an idea need have nothing to do with the so-called depth of its setting. Einstein remained for some time reluctant to frame his ideas in mathematics. He felt that the common use of mathematics was to dumbfound the reader. The dominant quality of connectedness is clarity. The more clear the idea, the more surely the connections strike home. It will be seen that the form of complexification, which progress has assumed, will have buried real progress in mathematics in too-narrow contexts beneath mounds of unnecessary arabesques. The false basis of mathematics must as surely harm the good work done as it propagates the poorly done work.

Mathematics is not apart from the only world. And that world is, for us, the connectedness of wholesome ideas, to use Van Gogh's adjective. These ideas are only to be found in the minds of mathematicians. Mathematics is only the activity of such ideas in the minds that reach beyond explanation into representation.

Looking at the history of mathematics, we see that reference is a good indicator of seminal work and ideas. And an accompaniment of reference is resistance. If an idea affects a large number of thinkers it is not because they were idly waiting for it. They were already working with the accepted ideas and were often opposed to the new work when it appeared. Occasionally, the common thinking is predisposed to accept a new framework, such as relativity or evolution, and opposition is overcome more quickly. But this predisposition is not a validation of the ideas. We are often predisposed to accept the incorrect idea.

One will not move long among mathematicians without hearing them justify their research in the following way: They will say that the research may not be connected to anything but that we never know what will turn out to be important. There are a couple of problems with this approach. For starters, if we do not know what we are searching for, we are not searching. And so such activity cannot be research, although it may entail searching for more grant money. Such work is best described as employment. Further, and more importantly, we do know what is important to humans. If we do not know what work is important, to continue in such work shows our moral lassitude. To insist that we do not know what is important to us is to imply that we are working for something other than each other. And just what would that something be? (As if we were working for some alien race whose need we could not fathom.)

Pure mathematics is currently like linguistics. Both claim to be true with regard to absolute form apart from any reference. But as one rises over tier upon tier of abstraction, one must stop at the point where there is still some object referenced by each abstraction. For the only aim of mathematics can be the understanding of the only world. Abstraction beyond reference is arbitrary and easily passes beyond the bounds of meaning, as our study of antinomy has shown. While it is true that a formal language will operate upon the arbitrary, that arbitrariness does not arise in a vacuum. It arises from preconception and belief, assumption and desire. It reveals a medieval and unreasoning mode of thought. And such a basis invariably establishes a structure of authority to which it may appeal rather than a foundation of reason and understanding.

Influenced by a century of formalism, understanding in mathematics has been largely reduced to the comprehension of evermore complex structures of tautologies. An example of this is the recent proof that claims to affirm Fermat's last theorem. This proof is said to require a mathematician in the field from which the proof has arisen about a month to work through. I would like to suggest that understanding may, even in mathematics, takes another form.

Fermat claimed that he had a proof of his famous theorem but wrote that it would not fit into the margin of the book he was scribbling in. He was a sober old judge and we really have no reason to think he was pulling our leg. He clearly earned his place in mathematics through real understanding. His ongoing work was esteemed by men such as Pascal. And we recall that the mathematics available to Fermat composed a body of work easily within our understanding today.

Fermat's mathematical methods were concerned with the infinite and are sometimes referred to as the method of infinite descent. They seem to have been simple methods, which have not been preserved in detail, due to lack of exposition on his part. The more one looks at alleged expositions of his methods, the less it seems we know what his relation to mathematics had been, that is, what standpoint he took toward the objects of number.

What we lack, and what Fermat demonstrated, was a quality of understanding. Fermat's proofs as a whole were not infallible. But there is no reason to assume that he did not have this last proof correctly formed in his thought. If so, he accomplished it with only the mathematics available to him and it arose from a standpoint of understanding, the structure of which we have yet to regain. Such a standpoint and such understanding is not a measure of horsepower, so to speak. We lack, not the mental power, but the mental clarity.

The hagiography of genius, especially in mathematics, has led us into misunderstanding what we term brilliance or genius. It was not necessary that Newton be a superior intelligence by some empirical measure, by volume, so to speak. It was only necessary that his standpoint be far enough outside that of his contemporaries for it to have been obscure to them. And it needed to have been a fruitful one for it to have led to the results it did. The arrogance of accomplished genius is too often mistaken for the clarity of thought that is accomplished by standpoint alone. Rather than the metaphor of strength and victory, the standpoint that expresses itself in richly meaningful work is better likened to independence and altitude. Those who achieve such a standpoint often express humility before God and nature, although they may be insufferably proud among men.

This humility arises from the realization that every true idea is in itself a continuum. Any state of affairs that may be understood, may be infinitely better understood. The standpoint of clarity and the altitude of independent thought reveal to us our intimate relation to the zero distance, the decimal point, in all that can be understood. Any thinker able to contribute substantial, unifying and clear work will know that his work is doomed to fall by the next remapping of understanding by a higher standpoint.

The thrust of this critique is not merely to show that the antinomies of reason arise from the form of number. Nor is it a claim that recognizing the necessity of reference will alone repair the damages of formalism. We must see that mathematics must progress not by accretion but through the unfolding of our understanding of the only world.

We can begin by seeing that structures like the real number continuum, Euclidean space, and manifolds arise from the laws and principles of their underlying forms. The whole thrust of nineteenth century analysis that culminated in the programms of Russell, Hilbert and Brouwer was an anachronism. It was entirely motivated by the Hellenistic imperative of construction. We did not give up the compass and ruler. We remade them, infinitesimal and abstract, and renamed them. We have yet to see the victory of our thinking over magnitude, prematurely heralded by Spengler. Too much of our thinking revolves around complicated structures built with Greek tools upon the scale of Newton's fluxion.

Mathematical thought has glaciated. For a century it has produced countless fragments of little articles that are ground to dust in periodical rooms and pass from thought beneath the weight of time. Too much of mathematical activity has had little to do with mankind and thus little to do with mathematics.

We must find a basis for mathematics that is satisfied by understanding rather than by construction. No set of axioms or other mathematical objects can explain or uphold a mathematic. Each mathematic rests upon the demonstration of understanding. Each mathematical object is subject to principle. Wherever each object arises in its context, it must maintain its identity and obey the laws applicable to its setting. When we begin to make exceptions in the expression of a mathematical object, it is clear that we do not understand the object and its principles. And our work ceases to be mathematical.

Wittgenstein wrote:

> The point of intersection of two curves is not the common membership of two classes of points, it is the meeting of two laws.
> Philosophical Grammar, p.461
> Ludwig Wittgenstein, Univ. of Ca., 1978.

We must learn to understand and rely upon principle. If a contradiction arises in our mathematic, if a gap appears, we are not better off for having set patches of axioms upon every little breach. It is a clearer understanding, not necessarily a deeper one, that destroys contradiction and closes the gaps in our understanding.

Mathematical thought is often like an optical illusion that refuses to resolve itself to a particular standpoint. We must choose the higher standpoint. And the selfless choice is that of law, principle, and their consequences: clarity, consistency, and unity.

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Austin, Texas
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