On Niemand's Factoring of Polynomials

Professor Niemand

Late one night, recently, I was considering the problem of reducing this to lowest terms:

$$\frac{x^4+2x^3-10x^2-11x-12}{x^4+3x^3-7x^2-21x-36}$$

I had not intended to think about this problem at all. I happened to see it in passing and the sight of it was upsetting to me. I felt as if I should be able to solve it easily. But no ideas came. I further made the mistake of copying it out of Todhunter's Algebra onto my scratch paper. And there it sat, filling me with darkness and doubt – I said it was late.

It was at this moment that a cloud of tobacco smoke filled the doorway, and not the smoke of a blend that should be smoked indoors. From out of this cloud came the bowl of an immense Meerschaum pipe, followed by its smoker -- a large, bearded man in an ill-fitting suit. I recognized him immediately, just as you would.

It was Herr Niemand. We've heard of him all our lives. He is smarter than Newton (or Gauss). He can square the circle or trisect an angle with only a straight-edge and compass. And he regularly derives the roots of polynomials of arbitrary degrees greater than fourth. Just the man I wanted to see.

“Herr Niemand,” I said. “Welcome. Come in. What brings you here?”

“You appear to be struggling,” he replied. “And the sufferings of lesser mathematicians are pleasing to me.”

Nobody ever said he was a nice man.

“I am struggling,” I said and showed him my problem.

“Kinderspiel,” he shrugged. “I see no difficulty here at all.”

“Enlighten me,” I replied.

“Very well,” he said. “What is a polynomial?”

On Polynomials

“An algebraic expression?” I replied.

“Mein Gott,” he rumbled, “and a man is a hairless biped which does not lay eggs.”

“I have hair,” I said.

“Less and less,” he said, gesturing at my receding hairline. “Next you will tell me you lay eggs. What is a polynomial?”

“A series of values?” I ventured.

“Only from the vantage point of eternity. For mere mortals?”

“A value depending on x, then,” I said.

“Very good, mein Kind,” he said. “And of what nature is that value?”

“A number?”

“You are slow, but almost there,” he said. “As you look at these polynomials, this $x^4+3x^3-7x^2-21x-36$, what does it remind you of?”

“Chaos and old night?” I said, yawning.

“Am I boring you?” he thundered.

“No, mein Herr,” I answered promptly. “What should it remind me of?”

“Think back to arithmetic,” he said. “You recently read Herr De Morgan's Elements of Arithmetic, did you not? Think at least that far back.”

I did as instructed.

“I suppose it could remind me of De Morgan's explanation of our number system. Thirty-six ones, twenty-one tens, seven hundreds. And then you…”

“Excellent,” said Niemand. “And therefore?”

“Alright,” I said. “This could be a number. But the x's, the plus and minus signs, and the double digits confuse the issue. And how is this a particular number if we don't choose a value for x?”

“How indeed,” he replied. “Let us take a simpler example.”

When All X's are Tens

“Consider the following,” said the learned professor: “

$$\frac{x^3+3x^2+3x+2}{3x^3+8x^2+5x+2}$$

Now there are no tricky minus signs or double digits to confuse your tiny mind. If we proceed with your remembrance of Herr De Morgan's arithmetic lesson, what is the value of x?”

“Ten,” I said, for once sure of myself.

“You begin to show promise,” smiled the doctor. “And the fraction becomes?”

“

$$\frac{1332}{3852}$$

,” I said.

“Correct. Factor, please,” he commanded.

I factored, arriving at:

$$\frac{2^2\ 3^2\ 37}{2^2\ 3^2\ 41}$$

and showed Herr Niemand my result.

“Yes. Very good. And thus the Greatest Common Measure of these numbers?” he said.

“36,” I replied.

Again I was pretty sure of myself.

“Now restore your x,” he said, with a tone of our having arrived.

Apparently, I was still I transit.

“My what?” I asked.

“You are slow, mein Kind,” he said, “and thick. But we overlook your failings. In our discussion so far, what is x?”

“Ten?”
I was beginning to doubt even the obvious.

“Yes. Of course. So restore it,” he said. “Restore it to 36.”

“3x+6?” I said, slowly.

“There may be hope for you yet,” he said. But he shook his head doubtfully.

On the Nature of Factors

“Let us proceed,” said Herr Niemand. “Please divide your polynomials by 3x+6.”

I did so:

$$\frac{x^2+ \quad x+}{x^2+ \quad x+}$$

“It divides evenly,” I said.

“Of course,” he replied. “Numbers are numbers. Factors are factors.”

“But I'm not crazy about the fractional coefficients,” I said.

“Consider your divisor,” he said, pointing to my notes. “How many factors is that really?”

“Two,” I said. “3 and x+2.”

“True,” he said. “and if (3)(x+2) is a factor, then (n)(x+2) is a factor, so to speak. The n merely varies the fractional coefficients.”

“So the actual factor here is x+2,” I said.

Light began to shine in the darkness and I redivided:

$$\frac{x^2+x+1}{3x^2+2x+1}$$

“Yes,” he said, smiling and relighting his pipe.

“Then we can do it again,” I said.

“Do what again?” he asked.

“Use the same process to derive the roots of these equations,” I said.

“You can try,” he said. “Proceed.”

I proceeded.

$$\frac{111}{321} = \frac{3\ 37}{3\ 107}$$

“We have a problem,” I said. “The numerator gives us $3\ 3x+7$ and the denominator is $3\ x^2+7$. That's $9x+21$ and $3x^2+21$.”

“Which gives you 111 and 321,” he said. “Where is your problem?”

“What about the roots?” I asked.

He shook his head as if pitying me.

“Liebes Kind,” he said, “the numbers 37 and 107 are prime. And until you can divide them evenly…”

“And you can?” I asked.

And then I remembered.

“Of course, you can,” I said.

He smiled his irritating smile.

“You could teach me,” I said, hopefully.

“I tried that once,” he replied. “A Herr Galois, as I recall. It ended badly.”

“But he couldn't do that with primes,” I said.

“He could,” said Niemand. “I taught him. Unfortunately, as he was making his notes, his mind was pre-occupied with self-destruction. Such is not conducive to mathematical exposition.”

I thought a moment.

“So if it hadn't been for the French Academy's obtuseness…?” I said.

Niemand shrugged.

“Let us return to your original problem,” he said. “The witching hour approaches.”

On Those Troublesome Minus Signs

I wrote the problem down afresh:

$$\frac{x^4+2x^3-10x^2-11x-12}{x^4+3x^3-7x^2-21x-36}$$

"Proceed," said Niemand again.

"What about the minus signs?" I asked.

"Have you forgotten your subtraction?" he asked. "Arithmetic is arithmetic."

"You speak as if everything were number," I mused.

"And everything we encounter here is indeed just that," he said.

"You are a Pythagorean then?" I said, trying not to laugh.

"Certainly, not," he said. "My sense was mathematical, not metaphysical. Consider hyperbolic spaces, which have been shown to be closely related to irrational numbers. Indeed, what are they but a restatement of the irrationals in Byzantine form? Upon a spherical space, all lines are fractions of unity and so we restate the rationals in space. Euclid then becomes a restatement of the natural numbers."

"But arcs of a meridian or lines in the plane can be incommensurables," I protested.

"Incummensurable to what?" he asked. "And therefore commensurable to what? But time presses. Proceed with out method."

I finally did the math and wrote the following:

$$\frac{10878}{10254} = \frac{2\ 3\ 7^2\ 41}{2\ 3\ 7^2\ 37}$$

"Consider your Greatest Common Measure," said Niemand.

"That would be 294," I said. "Or $2x^2+9x+4$. But that won't work. Because we're left with, say, 41 or $4x+1$. And multiplying those two expressions gives us…"

"$8x^3+38x^2+25x+4$," he said. "Arithmetically correct. But not very helpful, as you can see. Consider your factors and look for one that will not give you fractional coefficients…"

"And whose last element is a factor of 12 and 36," I said.

“You do begin to show promise,” he smiled. “In the earlier problem, 18 was a factor. But x+8 would not have worked because 8 is not a factor of 2.”

“But neither was 6. Both 6 and 8 are multiples of 2,” I said. “3x+6.”

“Yes,” he said. “But you can factor x+2 out of that GCM. That cannot be said of x+8.”

“I see,” I said. “So here given 2 3 7^2 we have factors of 6, 14, 21, and 49. Fourteen should work. So factoring out x+4 gives us

$$\frac{x^3-2x^2-2x-3}{x^3-x^2-3x-9}$$

and that's as far as we can go. Because now we're down to 3 7 41 and 3 7 37. Or 2x+ 1 times either 4x+1 or 3x+7. Both give second degree results.”

The Limits of Mortality

I sat back in my chair.

“So these are now in lowest terms?” I asked

“For you, perhaps,” he said. “I would offer to finish it for you. But you wouldn't be able to write down the incommensurables. For a mere mortal, you certainly have…”

He was cut off by the clock striking midnight and disappeared, leaving only his unpleasant smoke behind. I wondered what it was he thought I had. Then I decided I'd probably be happier not knowing. At least I had grasped the basics of his heuristic. Niemand's Method can be used more broadly than factoring polynomials. For instance, you can find the square roots of polynomials. As simple examples:

$$x^2+2x+1 \text{ is } 121$$
$$\sqrt{121} = 11 \text{ or } x+1$$

$$x^2-2x+1 \text{ is } 81$$
$$\sqrt{81} = 9 = 10 - 1 \text{ or } x-1$$

In the case square roots, this just shows the basic reasoning. So if you were to want the square root of:

$$4x^4-12x^3+5x^2+6x+1 \text{ or } 28561$$
$$\sqrt{28561} = 169 \text{ or } x^2+6x+9$$

Not helpful. But you would realize that the first term must be $2x^2$ (because it has to square into $4x^4$) and the last term must be ±1 (because none of the coefficients smell like nines). So reasoning from the options 201 and 199, the middle term is, from the latter, clearly -3x and the result:

$$2x^2-3x-1 = 169$$

You can see that all of this is simply arithmetic combined with some extra thought to handle the minus signs. It would be nice if, somehow, Niemand's Method took us further and factored the “prime” polynomials. But as a mere mortal, I'll take what I can get.

Niemand's Standpunkt

It does strike me that something fundamental is going on here. Our civilization's mathematic, in the largest sense, is base ten. In these integral functions, or polynomials, the coefficients are base ten. So making x=10 seems somehow significant.

You can see in the last example that 28561 is attainable with infinitely many combinations of coefficients using positive or negative terms as needed. Then isn't $2x^4+8x^3+5x^2+6x+1$ somehow a more fundamental representation of $4x^4-12x^3+5x^2+6x+1$ in some sense?

I am aware that this rather turns our views of functions on its head. Since the advent of analytical geometry, calculus, and function theory, we look at a polynomial like this as a variable y which, as a function of x, takes on infinitely many values. Further, the function, so to speak, is its graph and vice versa. From this point of view, those infinite polynomials which at x=10 converge at 28561 are an infinity of spaghetti with a common intersection which they consider infinitely insignificant.

But in Niemand's view, this infinite set of functions would be the class that maps to 28561 of which $2x^4+8x^3+5x^2+6x+1$ is the fundamental representation. No other value of x, so far as I can see, is “fundamental” in this sense. If we made x=6, we'd be talking base six and would have to convert the coefficients -- if not go live in another civilization entirely. This seems to show a fundamental element in Niemand's approach given the form of our mathematic.

This gives rise to such questions as when x^2-2x+1, being 81, is somehow at bottom 8x+1, then what is the relation that must arise from a line and connect upwards into various curves? And what about the classes of constants?

What that element is would require perhaps a deeper basis of number theory and function theory than I currently have. And while more function theory is on my to-do list, I feel the same way about number theory that I do about many green vegetables. My aversion is foolish and irrational. But there you are.

So should someone with more talent and better tools actually take this somewhere, I expect them to give Herr Doktor Niemand due credit. I am indebted to Lewis Carroll for introducing me to him in Carroll's **Euclid and His Modern Rivals**. And nobody deserves more respect than Niemand does.

Binomial Theorem

I was thinking about Niemand while working with Newton's Binomial Theroem. And in this case, Niemand remains an heuristic rather than a general solution. If we take

$$(x+2)^2$$

and use Niemand's x=10, 12^2 is 144 or

$$x^2+4x+4$$

which is the correct binomial expansion. But for

$$(x+2)^3$$

we have $12^3 = 1728$ and

$$x^3+7x^2+2x+8$$

is not the binomial expansion. But knowing the **form** of the binomial expansion, we know that the coefficient of unity for x^3 and 8 are correct. And we know that the second coefficient must be divisible by 2 and the third by 3. So taking one from the hundred's place and putting ten in the ten's place, we have

$$x^3+6x^2+12x+8$$

which is correct. As a final example, let's take

$$(x+3)^3$$

$13^3 = 2197$ or

$$2x^3+x^2+9x+7$$

If we fix the outer ones first, we have

$$x^3+11x^2+7x+27$$

which we can think of as

$$x^3+ax^2+bx+27$$

where a must be divisible by 3 and b by 9. So moving 2 from the 100s into the tens, we have

$$x^3+9x^2+27x+27$$

which is again the correct expansion.

What so far keeps me thinking that Niemand's method is a heuristic is that it does not express things generally. It's specific expression is, however, a useful tool. If in $(x+3)^3$ we let x=2, we get 125 or x^2+2x+5 and one cannot work backwards from this to $x^3+9x^2+27x+27$. I want to say that Niemand's approach is a slice through everything in base 10. So if x=2 we are somehow working with base 2 using base 10 (or vice versa, I'd have to think about it to be sure) and the result is a hairball.

Niemand's method is useful if and only if we know the **form** of things. Then, because of, in a sense, everything being number, what is true of Niemand is true of the binomial expansion or the factors of a polynomial. But we must know the true form of those things in order to use Niemand. And then his method greatly simplifies what we are doing in the same way that the elementary division algorithm simplifies division or using Euclid's algorithm simplifies creating a continued fraction. (pre 10jan18)

Niemand Values

For convenience, among other things, I now refer to the integer given by Niemand's method as the nieval of a polynomial. Further, if we make x=10,
we get the nieval. If x=-10, we get the -nieval. Any other nieval must reveal its x, such as nieval(3) where x=3. These were adopted when the following occurred to me.

It first occurred to me that if we use x=-10 as well as x=10, we get a second base 10 result. And the factors of both must reveal the same roots of the equation. Take

$$x^2+5x+6$$

which is (x+2)(x+3). We have nieval=156, factors 2,2,3,13 and -nieval=56, factors 2,2,2,7. From the first we get 12, 13: (x+2)(x+3); from the second, 8, 7: (x-2)(x-3). I haven't discovered the law for this use of -10 yet. But it is as if it yields instead (-x+2)(-x+3) where -x is then denoted x, and again we get: (x+2)(x+3). This is clearly **not** mathematics. But it will be when I figure out what is going on here. So don't mind my confusion.

Leaving that process as giving results but creating a metaphysical muddle, it next occurred to me that we could use our two nievals to find the roots of a quadratic (although an easier method already exists). Let the unknown roots be a,b and we have:

$$(10-a)(10-b) = \text{nieval}$$

$$(-10-a)(-10-b) = -nieval$$
or
$$100-10a-10b+ab = nieval$$
$$100+10a+10b+ab = -nieval$$

First note that in x^2+px+q, $ab = q$. So everything is given but a and b and we have two equations and two unknowns. It then occurred to me that one could use nieval(1), nieval(2), and nieval(3) to get three equations with three unknowns for a cubic. I conjecture that this could be used for any degree of equation as long as we could fill in the blanks as in ab above.

I can't see why it wouldn't work for any degree and would go after a fifth degree equation myself for the possible glory of it. But the muddle of resulting coefficients exceeds my current skill with symmetrical equations of the roots. When my Theory of Equations and my use of summation signs in algebra is further along, and if I don't in the meantime discover a reason why this wouldn't work (in spite of proofs of impossibility), I'll give it a go. Proven impossibilities shouldn't scare us. Sometimes we just have to re-prove them on our own. You caught the double entendre, non? (10jan18)

Of Prime Factors and the Factors of Primes

Sometimes the nieval is a prime number like 113. I was hoping that this would tell us something about the roots. But the nieval doesn't seem to relate numerically to the roots. Then I looked to see if the sign of a prime nieval told us anything. And it doesn't. All that a prime nieval seems to indicate is that (most of) the roots are either real surds or complex root-pairs, Of course, there might be a root of unity in there as well depending on the degree of the function.

Of course, (do I say that too often?) a prime nieval does clue us into some things about the roots. A quadratic with a prime nieval has two complex roots or two surds. A cubic with a prime nieval has, of course, one real root and two complex. You can fiddle with Newton's rule, maybe build it into your TI programmable calculator, for higher degrees of equations. And what if a non-prime nieval has two prime factors? Well. again it depends on the function's dimension but it does clue you into what is actually there, depending on the circumstance. So primes are interesting in this space. But they aren't magical.

My disappointment here is always that I am looking for the law of roots, which must absolutely be locked somewhere inside the form of number. After making it through my first Theory of Equations text, it seems to me -- and I could be wrong -- that being able to compute the roots and graph the equations has led to the TofE being largely waste-binned. I'm sure all the people who **cared** about the actual value of roots have washed their hands of it. For my part, I can't bring myself to care what the value of a root is (although Horner's Method is cooler than I thought it would be). What I want is the key to all polynomials so that when I turn it, all the roots will spill out. The watchword remains: Keep the Faith and Dream On.

But all is not lost. If you pass almost magically into the mirror, like Alice, and look at these prime nievals from the other side, it does reveal an interesting truth. Let me explain. First pick an odd prime nieval, like our 113. Then for any n∈**N**, create a rational intergral function that produces 113 for its nieval. Clearly, arithmetic being what it is, you can always do this. The final coefficient, of x^0, will be an odd integer. And this odd integer will then be the product of at least n-1 roots. And because the nieval is prime, those roots are either surds or complex pairs. I believe what we can state is this:

∀n∈N, ∀odd z∈Z, z has n or n-1 factors, either surds or complex numbers.

Maybe everyone already knew this and I just missed that lecture. But it's pretty trippy to think about. Pick a natural number, any natural number. Say, a gazillion-and-26. Now pick any odd integer (not unity): 3, -7, whatever. And that 3 or -7 has a gazillion-and-26 factors, surd or complex. In fact, 3 or -7 or 27 or whatever has every number (or at least every even-number) of such factors. It's like, Cauchy, dude, check it out… (02feb18)